

AVERAGE VALUES OF L -FUNCTIONS IN EVEN CHARACTERISTIC

SUNGHAN BAE AND HWANYUP JUNG

ABSTRACT. Let $k = \mathbb{F}_q(T)$ be the rational function field over a finite field \mathbb{F}_q , where q is a power of 2. In this paper we solve the problem of averaging the quadratic L -functions $L(s, \chi_u)$ over fundamental discriminants. Any separable quadratic extension K of k is of the form $K = k(x_u)$, where x_u is a zero of $X^2 + X + u = 0$ for some $u \in k$. We characterize the family \mathcal{I} (resp. \mathcal{F} , \mathcal{F}') of rational functions $u \in k$ such that any separable quadratic extension K of k in which the infinite prime $\infty = (1/T)$ of k ramifies (resp. splits, is inert) can be written as $K = k(x_u)$ with a unique $u \in \mathcal{I}$ (resp. $u \in \mathcal{F}$, $u \in \mathcal{F}'$). For almost all $s \in \mathbb{C}$ with $\operatorname{Re}(s) \geq \frac{1}{2}$, we obtain the asymptotic formulas for the summation of $L(s, \chi_u)$ over all $k(x_u)$ with $u \in \mathcal{I}$, all $k(x_u)$ with $u \in \mathcal{F}$ or all $k(x_u)$ with $u \in \mathcal{F}'$ of given genus. As applications, we obtain the asymptotic mean value formulas of L -functions at $s = \frac{1}{2}$ and $s = 1$ and the asymptotic mean value formulas of the class number h_u or the class number times regulator $h_u R_u$.

1. INTRODUCTION

In *Disquisitiones Arithmeticae* [7], Gauss presented two famous conjectures concerning the average values of class numbers associated with binary quadratic forms over \mathbb{Z} , which can be restated as the average values of class numbers of orders in quadratic number fields. The imaginary case of these conjecture was first proved by Lipschitz, and the real case by Siegel [17]. By the Dirichlet's class number formula, these two conjectures can be stated as averages of the values of quadratic Dirichlet L -functions at $s = 1$. In [17], Siegel showed how to average over all discriminants. Let χ_d be the quadratic character defined by the Kronecker symbol $\chi_d(n) = (\frac{d}{n})$ and

$$L(s, \chi_d) = \sum_{n=1}^{\infty} \chi_d(n) n^{-s}$$

be the Dirichlet series associated to χ_d . Siegel [17] has obtained averaging formulas for $L(1, \chi_d)$ over all positive discriminants d between 1 and N , or all negative discriminants d such that $1 < |d| \leq N$. From these averaging formulas with Dirichlet's class number formula, Siegel has obtained averaging formulas for the class number h_d or the class number times regulator $h_d R_d$ over all positive discriminants d between 1 and N , or all negative discriminants d such that $1 < |d| \leq N$. At the critical point $s = \frac{1}{2}$, Jutila [14] derived an asymptotic formula for

$$\sum_d L^k(\tfrac{1}{2}, \chi_d) \quad (k = 1, 2),$$

where d runs over fundamental discriminants in the interval $0 < d \leq X$. In [8], using the Eisenstein series of $\frac{1}{2}$ -integral weight, for $s \in \mathbb{C}$ with $\operatorname{Re}(s) \geq 1$, Goldfeld and Hoffstein obtained an asymptotic formula for $\sum L(s, \chi_m)$, where the sum is either over positive square-free m between 1 and N , or over negative square-free m with $1 < |m| \leq N$. Putting $s = 1$ and using the Dirichlet's class number formula, one can average the class number h_m or the class number times regulator $h_m R_m$ over the fields $\mathbb{Q}(\sqrt{m})$.

Now, we introduce the analogous results in function fields over finite fields. Let $k = \mathbb{F}_q(T)$ be the rational function field over a finite field \mathbb{F}_q of q elements, and $\mathbb{A} = \mathbb{F}_q[T]$ be the ring of

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polynomials. First, we consider the case of q being odd. Let χ_N be the quadratic character defined by the Kronecker symbol $\chi_N(f) = (\frac{N}{f})$ and

$$L(s, \chi_N) = \sum_{\substack{f \in \mathbb{A} \\ f: \text{monic}}} \chi_N(f) |f|^{-s}$$

be the quadratic Dirichlet L -function associated to χ_N . In [10], Hoffstein and Rosen has obtained a result on averaging $L(1, \chi_N)$ over all non-square monic polynomials $N \in \mathbb{A}$ (all discriminants) of given degree. Using the averaging of $L(1, \chi_N)$ with the Dirichlet's class number formula, they solved the problem of averaging the class number h_N or the class number times regulator $h_N R_N$ over all non-square monic polynomials $N \in \mathbb{A}$ of given degree. Moreover, by defining and analyzing metaplectic Eisenstein series in function fields, Hoffstein and Rosen [10] also succeed in averaging $L(s, \chi_N)$ over all square-free polynomials N (fundamental discriminants) of given degree. Andrade and Keating [2] and Jung [12] have obtained asymptotic formulas for $\sum L(\frac{1}{2}, \chi_N)$, where the sum is over all monic square-free N of given degree, by elementary analytic method using approximate functional equation. Recently, Andrade [1] and Jung [13] also have obtained asymptotic formulas for $\sum L(1, \chi_N)$.

In the case of q being even, Chen [5] obtained formulas of average values of L -functions associated to orders in quadratic function fields, and then derived formulas of average class numbers of these orders. The aim of this paper is to solve the problem of averaging $L(s, \chi_u)$ over fundamental discriminants in even characteristic. Any separable quadratic extension K of k is of the form $K = k(x_u)$, where x_u is a zero of $X^2 + X + u = 0$ for some $u \in k$. We characterize three families \mathcal{F} , \mathcal{F}' and \mathcal{I} of rational functions $u \in k$ such that any separable quadratic extension K of k can be written uniquely as $K = K(x_u)$, where $u \in \mathcal{F}$, $u \in \mathcal{F}'$ or $u \in \mathcal{I}$ according as the infinite prime $\infty = (1/T)$ of k splits, is inert or is ramified in K (see §2.2). By extending the analytic methods in [1–4, 12, 13, 16], for almost all $s \in \mathbb{C}$ with $\text{Re}(s) \geq \frac{1}{2}$, we obtain the asymptotic formulas for the summation of $L(s, \chi_u)$ over all K_u with $u \in \mathcal{F}$, all K_u with $u \in \mathcal{F}'$ or all K_u with $u \in \mathcal{I}$ of given genus (see Theorem 2.4). In [1, 2, 12, 13], the authors have obtained the asymptotic formulas for the summation of $L(s, \chi_N)$ only at $s = \frac{1}{2}$ and $s = 1$. The asymptotic formulas for the summation of $L(s, \chi_u)$ obtained in this paper hold for almost all $s \in \mathbb{C}$ with $\text{Re}(s) \geq \frac{1}{2}$. This can be regarded as an analogue of the result of Hoffstein and Rosen [10] in even characteristic. The method of proving “Approximate” functional equations of L -functions (Lemma 3.1) in this paper also holds in odd characteristic case and all results in §3.2 hold in odd characteristic too. Thus our method can also be applied to obtain the asymptotic formulas for the summation of $L(s, \chi_N)$ for almost all $s \in \mathbb{C}$ with $\text{Re}(s) \geq \frac{1}{2}$ in odd characteristic. The methods and calculations in [1, 2, 10] also can give the same asymptotic formulas for $\text{Re}(s) \geq \frac{1}{2}$. As applications, we obtain the asymptotic mean value formulas of L -functions at $s = \frac{1}{2}$ and $s = 1$ (see Corollaries 2.6 and 2.7), and using the Dirichlet's class number formula, we also obtain the asymptotic mean value formulas of the class number h_u or the class number times regulator $h_u R_u$ (see Theorem 2.8 and Corollary 2.9).

2. STATEMENT OF RESULTS

2.1. Some Background on $\mathbb{A} = \mathbb{F}_q[T]$. Let q be a power of 2. Let $k := \mathbb{F}_q(T)$ be the rational function field with a constant field \mathbb{F}_q , $\infty = (1/T)$ the infinite prime of k , and $\mathbb{A} := \mathbb{F}_q[T]$. We denote by \mathbb{A}^+ the set of monic polynomials in \mathbb{A} and by \mathbb{P} the set of monic irreducible polynomials in \mathbb{A} . Throughout this paper, any monic irreducible polynomial $P \in \mathbb{P}$ will be also called a “prime” polynomial. For a positive integer n we denote by \mathbb{A}_n^+ the set of monic polynomials in \mathbb{A} of degree n and by \mathbb{P}_n the set of monic irreducible polynomials in \mathbb{A} of degree n . The *norm* $|f|$ of a polynomial $f \in \mathbb{A}$ is defined to be $|f| := \#(\mathbb{A}/f\mathbb{A}) = q^{\deg(f)}$ for $f \neq 0$, and $|f| := 0$ for $f = 0$, where $\#X$ denotes the cardinality of a set X . For any $0 \neq f \in \mathbb{A}$, let $\Phi(f) := \#(\mathbb{A}/f\mathbb{A})^\times$, and let $\text{sgn}(f)$ be

the leading coefficient of f . Let $\wp : k \rightarrow k$ be the additive homomorphism defined by $\wp(x) = x^2 + x$.

The zeta function $\zeta_{\mathbb{A}}(s)$ of \mathbb{A} is defined for $\operatorname{Re}(s) > 1$ to be the following infinite series:

$$\zeta_{\mathbb{A}}(s) := \sum_{f \in \mathbb{A}^+} \frac{1}{|f|^s} = \prod_{P \in \mathbb{P}} \left(1 - \frac{1}{|P|^s}\right)^{-1}, \quad \operatorname{Re}(s) > 1. \quad (2.1)$$

It is well known that $\zeta_{\mathbb{A}}(s) = \frac{1}{1-q^{1-s}}$.

The monic irreducible polynomials in \mathbb{A} also satisfies the analogue of the Prime Number Theorem. In other words we have the following.

Theorem 2.1 (Prime Polynomial Theorem). *Let $\pi_{\mathbb{A}}(n)$ denote the number of monic irreducible polynomials in \mathbb{A} of degree n . Then, we have*

$$\pi_{\mathbb{A}}(n) = \#\mathbb{P}_n = \frac{q^n}{n} + O\left(\frac{q^{\frac{n}{2}}}{n}\right). \quad (2.2)$$

2.2. Quadratic function field in even characteristic. Any separable quadratic extension K of k is of the form $K = K_u := k(x_u)$, where x_u is a zero of $X^2 + X + u = 0$ for some $u \in k$. We say that $u, v \in k$ are equivalent if $K_u = K_v$. It is known that u and v are equivalent if and only if $u + v = \wp(w)$ for some $w \in k$ (see [9] or [11]). Fix an element $\xi \in \mathbb{F}_q \setminus \wp(\mathbb{F}_q)$. The following lemma is due to Y. Li. In fact, Y. Li obtained the result which holds for any Artin-Schreier extensions of the rational function fields of any characteristic.

Lemma 2.2. *Any separable quadratic extension K of k is of the form $K = K_u$, where $u \in k$ can be uniquely normalized to satisfy the following conditions:*

$$u = \sum_{i=1}^m \sum_{j=1}^{e_i} \frac{Q_{ij}}{P_i^{2j-1}} + \sum_{\ell=1}^n \alpha_{\ell} T^{2\ell-1} + \alpha, \quad (2.3)$$

where $P_i \in \mathbb{P}$ are distinct, $Q_{ij} \in \mathbb{A}$ with $\deg(Q_{ij}) < \deg(P_i)$, $Q_{ie_i} \neq 0$, $\alpha \in \{0, \xi\}$, $\alpha_{\ell} \in \mathbb{F}_q$ and $\alpha_n \neq 0$ for $n > 0$.

Proof. Since it is difficult to find the reference for it, we will give the proof due to Y. Li. We know that every element $u \in k$ can be uniquely written as

$$u(T) = \sum_{i=1}^m \sum_{e_{ij}=1}^{e_i} \frac{Q_{ij}}{P_i^{e_{ij}}} + \sum_{\ell=1}^n \alpha_{\ell} T^{\ell} + \alpha,$$

where $P_i \in \mathbb{P}$ are distinct, $\deg(Q_{ij}) < \deg(P_i)$, $Q_{ie_i} \neq 0$ and $\alpha_{\ell}, \alpha \in \mathbb{F}_q$. We can remove the term $\frac{Q_{ij}}{P_i^{e_{ij}}}$ with $2|e_{ij}$ as follows. Let $e_{ij} = 2k_{ij}$ and let $M_{ij} \in \mathbb{A}$ with $\deg(M_{ij}) < \deg(P_i)$ such that

$$M_{ij}^2 \equiv Q_{ij} \pmod{P_i}.$$

Then we have

$$\frac{Q_{ij}}{P_i^{e_{ij}}} + \left(\frac{M_{ij}}{P_i^{k_{ij}}}\right)^2 + \frac{M_{ij}}{P_i^{k_{ij}}} = \frac{Q_{ij} + M_{ij}^2}{P_i^{e_{ij}}} + \frac{M_{ij}}{P_i^{k_{ij}}}.$$

Similarly, we can remove even degree term $\alpha_{2\ell} T^{2\ell}$ and we get the desired form. Now it is clear that any two $u, v \in k$ which are normalized as in (2.3) are equivalent if and only if $u = v$. \square

Let $u \in k$ be normalized one as in (2.3). The infinite prime $\infty = (1/T)$ splits, is inert or ramified in K_u according as $n = 0$ and $\alpha = 0$, $n = 0$ and $\alpha = \xi$, or $n > 0$. Then the field K_u

is called *real*, *inert imaginary*, or *ramified imaginary*, respectively. The discriminant D_u of K_u is given by

$$D_u = \prod_{i=1}^m P_i^{2e_i} \quad \text{if } n = 0$$

and

$$D_u = \prod_{i=1}^m P_i^{2e_i} \cdot (1/T)^{2n} \quad \text{if } n > 0,$$

and, by the Hurwitz genus formula ([18, Theorem III.4.12]), the genus g_u of K_u is given by

$$g_u = \frac{1}{2} \deg(D_u) - 1. \quad (2.4)$$

For $M \in \mathbb{A}^+$, let $r(M) := \prod_{P|M} P$ and $t(M) := M \cdot r(M)$. For $P \in \mathbb{P}$, let ν_P be the normalized valuation at P , that is, $\nu_P(M) = e$, where $P^e \| M$. Let \mathcal{B} be the set of monic polynomials M such that $\nu_P(M) = 0$ or odd for any $P \in \mathbb{P}$, that is, $t(M)$ is a square, and \mathcal{C} be the set of rational functions $\frac{D}{M} \in k$ such that $D \in \mathbb{A}$, $M \in \mathcal{B}$ and $\deg(D) < \deg(M)$. For $M \in \mathcal{B}$, let $\ell_P := \frac{1}{2}(\nu_P(M) + 1)$ for any $P|M$. Also we let \mathcal{E} be the set of rational functions $\frac{D}{M} \in \mathcal{C}$ of the form

$$\frac{D}{M} = \sum_{P|M} \sum_{i=1}^{\ell_P} \frac{A_{P,i}}{P^{2i-1}},$$

where $\deg(A_{P,i}) < \deg(P)$ for any $P|M$ and for all $1 \leq i \leq \ell_P$. Note that for $\frac{D}{M} \in \mathcal{E}$, $\gcd(D, M) = 1$ if and only if $A_{P,\ell_P} \neq 0$ for all $P|M$. Let \mathcal{F} be the set of rational functions $\frac{D}{M} \in \mathcal{E}$ such that $A_{P,\ell_P} \neq 0$ for all $P|M$ and $\mathcal{F}' := \{u + \xi : u \in \mathcal{F}\}$. By the normalization in (2.3), we can see that $u \mapsto K_u$ defines a one-to-one correspondence between \mathcal{F} (resp. \mathcal{F}') and the set of all real (resp. inert imaginary) separable quadratic extensions of k . For any positive integer s , let \mathcal{G}_s be the set of polynomials $F(T) \in \mathbb{A}$ of the form

$$F(T) = \alpha + \sum_{i=1}^s \alpha_i T^{2i-1}, \quad \text{where } \alpha \in \{0, \xi\}, \alpha_i \in \mathbb{F}_q \text{ and } \alpha_s \neq 0.$$

Let $\mathcal{G} := \bigcup_{s \geq 1} \mathcal{G}_s$ and $\mathcal{I} := \{u + F : u \in \bar{\mathcal{F}} \text{ and } F \in \mathcal{G}\}$, where $\bar{\mathcal{F}} = \mathcal{F} \cup \{0\}$. By the normalization in (2.3), we can see that $w \mapsto K_w$ defines a one-to-one correspondence between \mathcal{I} and the set of all ramified imaginary separable quadratic extensions of k .

2.3. Hasse symbol and L -functions. Let $P \in \mathbb{P}$. For any $u \in k$ whose denominator is not divisible by P , the Hasse symbol $[u, P]$ with values in \mathbb{F}_2 is defined by

$$[u, P] := \begin{cases} 0 & \text{if } X^2 + X \equiv u \pmod{P} \text{ is solvable in } \mathbb{A}, \\ 1 & \text{otherwise.} \end{cases}$$

For $N \in \mathbb{A}$ prime to the denominator of u , write $N = \text{sgn}(N) \prod_{i=1}^s P_i^{e_i}$, where $P_i \in \mathbb{P}$ are distinct and $e_i \geq 1$, and define $[u, N]$ to be $\sum_{i=1}^s e_i [u, P_i]$.

For $u \in k$ and $0 \neq N \in \mathbb{A}$, we also define the quadratic symbol:

$$\left\{ \frac{u}{N} \right\} := \begin{cases} (-1)^{[u, N]} & \text{if } N \text{ is prime to the denominator of } u, \\ 0 & \text{otherwise.} \end{cases}$$

This symbol is clearly additive in its first variable, and multiplicative in the second variable.

For the field K_u , we associate a character χ_u on \mathbb{A}^+ which is defined by $\chi_u(f) = \left\{ \frac{u}{f} \right\}$, and let $L(s, \chi_u)$ be the L -function associated to the character χ_u : for $s \in \mathbb{C}$ with $\text{Re}(s) \geq 1$,

$$L(s, \chi_u) := \sum_{f \in \mathbb{A}^+} \frac{\chi_u(f)}{|f|^s} = \prod_{P \in \mathbb{P}} \left(1 - \frac{\chi_u(P)}{|P|^s} \right)^{-1}.$$

It is well known that $L(s, \chi_u)$ is a polynomial in q^{-s} . Letting $z = q^{-s}$, write $\mathcal{L}(z, \chi_u) = L(s, \chi_u)$. Then, $\mathcal{L}(z, \chi_u)$ is a polynomial in z of degree $2g_u + \frac{1}{2}(1 + (-1)^{\varepsilon(u)})$, where $\varepsilon(u) = 1$ if K_u is ramified imaginary and $\varepsilon(u) = 0$ otherwise. Also we have that $\mathcal{L}(z, \chi_u)$ has a “trivial” zero at $z = 1$ (resp. $z = -1$) if and only if K_u is real (resp. inert imaginary), so we can define the “completed” L -function as

$$\mathcal{L}^*(z, \chi_u) := \begin{cases} \mathcal{L}(z, \chi_u) & \text{if } K_u \text{ is ramified imaginary,} \\ (1-z)^{-1} \mathcal{L}(z, \chi_u) & \text{if } K_u \text{ is real,} \\ (1+z)^{-1} \mathcal{L}(z, \chi_u) & \text{if } K_u \text{ is inert imaginary,} \end{cases} \quad (2.5)$$

which is a polynomial of even degree $2g_u$ satisfying the functional equation

$$\mathcal{L}^*(z, \chi_u) = (qz^2)^{g_u} \mathcal{L}^*\left(\frac{1}{qz}, \chi_u\right). \quad (2.6)$$

2.4. Main results. For $M \in \mathcal{B}$, let

$$\tilde{M} := \prod_{P|M} P^{(\nu_P(M)+1)/2} = \sqrt{t(M)}.$$

For positive integer n , let

$$\begin{aligned} \mathcal{B}_n &:= \{M \in \mathcal{B} : \deg(t(M)) = 2n\}, \quad \mathcal{C}_n := \left\{ \frac{D}{M} \in \mathcal{C} : M \in \mathcal{B}_n \right\}, \\ \mathcal{E}_n &:= \mathcal{E} \cap \mathcal{C}_n, \quad \mathcal{F}_n := \mathcal{F} \cap \mathcal{E}_n, \quad \mathcal{F}'_n := \{u + \xi : u \in \mathcal{F}_n\}. \end{aligned}$$

Under the above correspondence $u \mapsto K_u$, \mathcal{F}_n (resp. \mathcal{F}'_n) corresponds to the set of all real (resp. inert imaginary) separable quadratic extensions K_u of k with genus $n-1$.

Let $\mathcal{F}_0 = \{0\}$. For any integers $r \geq 0$ and $s \geq 1$, let $\mathcal{I}_{(r,s)} = \{u + F : u \in \mathcal{F}_r, F \in \mathcal{G}_s\}$. For any integer $n \geq 1$, let \mathcal{I}_n be the union of all $\mathcal{I}_{(r,s)}$, where (r,s) runs over all pairs of nonnegative integers such that $s > 0$ and $r + s = n$. Then, under the correspondence $u \mapsto K_u$, \mathcal{I}_n corresponds to the set of all ramified imaginary separable quadratic extensions K_u of k with genus $n-1$.

Lemma 2.3. *For a positive integer n , we have $\#\mathcal{B}_n = q^n$, $\#\mathcal{E}_n = q^{2n}$, $\#\mathcal{F}_n = \zeta_{\mathbb{A}}(2)^{-1} q^{2n}$ and $\#\mathcal{I}_n = 2\zeta_{\mathbb{A}}(2)^{-1} q^{2n-1}$.*

Proof. The map $\mathcal{B}_n \rightarrow \mathbb{A}_n^+$ defined by $M \mapsto \tilde{M}$ and the map $\mathbb{A}_n^+ \rightarrow \mathcal{B}_n$ defined by $N \mapsto N^* := N^2/r(N)$ are inverse to each other. Thus we have $\#\mathcal{B}_n = \#\mathbb{A}_n^+ = q^n$. For each $M \in \mathcal{B}_n$, there are q^n (resp. $\Phi(\tilde{M})$) D 's such that $\frac{D}{M} \in \mathcal{E}_n$ (resp. $\frac{D}{M} \in \mathcal{F}_n$). Thus $\#\mathcal{E}_n = q^n \cdot \#\mathcal{B}_n = q^{2n}$ and

$$\#\mathcal{F}_n = \sum_{M \in \mathcal{B}_n} \Phi(\tilde{M}) = \sum_{\tilde{M} \in \mathbb{A}_n^+} \Phi(\tilde{M}) = \zeta_{\mathbb{A}}(2)^{-1} q^{2n}$$

by [15, Proposition 2.7]. Since $\#\mathcal{G}_s = 2\zeta_{\mathbb{A}}(2)^{-1} q^s$ for $s \geq 1$, we have

$$\#\mathcal{I}_n = \sum_{s=1}^n \#\mathcal{I}_{(n-s,s)} = \sum_{s=1}^n \#\mathcal{F}_{n-s} \cdot \#\mathcal{G}_s = 2\zeta_{\mathbb{A}}(2)^{-1} q^{2n-1}.$$

□

For any $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 0$, let

$$P(s) = \prod_P \left(1 - \frac{1}{|P|^s(|P|+1)} \right).$$

For an arbitrary small $\varepsilon > 0$, let X_ε be the set of complex numbers $s \in \mathbb{C}$ such that $\frac{1}{2} \leq \operatorname{Re}(s) < 1$ and $|s - \frac{1}{2}| > \varepsilon$, and $\bar{X}_\varepsilon = \{\frac{1}{2}\} \cup X_\varepsilon \cup \{s \in \mathbb{C} : \operatorname{Re}(s) \geq 1\}$. For the first moment of Dirichlet L -functions at $s \in \mathbb{C}$ with $\operatorname{Re}(s) \geq \frac{1}{2}$, we have the following theorem.

Theorem 2.4. *Let $s_1 = \frac{1}{2}(1 + \log_q 2) \leq 1$.*

(1) For an arbitrary small $\varepsilon > 0$ and $s \in \bar{X}_\varepsilon$, we have

$$\sum_{u \in \mathcal{I}_{g+1}} L(s, \chi_u) = 2\alpha_g(s)q^{2g+1} + \begin{cases} O(g2^{\frac{g}{2}}q^{(2-s)g}) & \text{if } \operatorname{Re}(s) < s_1, \\ O(gq^{\frac{3g}{2}}) & \text{if } \operatorname{Re}(s) \geq s_1, \end{cases}$$

where $\alpha_g(\frac{1}{2}) = \frac{P(1)}{\zeta_{\mathbb{A}}(2)}(g+1 + \frac{2}{\log q} \frac{P'}{P}(1))$, $\alpha_g(s) = \frac{\zeta_{\mathbb{A}}(2s)}{\zeta_{\mathbb{A}}(2)}P(2s)$ for $\operatorname{Re}(s) \geq 1$, and, for $s \in X_\varepsilon$,

$$\alpha_g(s) = \frac{\zeta_{\mathbb{A}}(2s)}{\zeta_{\mathbb{A}}(2)} \left\{ P(2s) - q^{(1-2s)(g+1)}P(2-2s) \right. \\ \left. + P(1) \left(q^{(1-2s)(g-\lfloor \frac{g-1}{2} \rfloor)} - q^{(1-2s)(\lfloor \frac{g}{2} \rfloor + 1)} \right) \right\}.$$

(2) For an arbitrary small $\varepsilon > 0$ and $\delta > 0$ and for $s \in \bar{X}_\varepsilon$ with $|s-1| > \delta$ or $s = 1$, we have

$$\sum_{u \in \mathcal{F}_{g+1}} L(s, \chi_u) = \beta_g(s)q^{2g+2} + \begin{cases} O(2^{\frac{g}{2}}q^{(2-s)g}) & \text{if } \operatorname{Re}(s) < s_1, \\ O(gq^{\frac{3g}{2}}) & \text{if } \operatorname{Re}(s) \geq s_1, \end{cases} \quad (2.7)$$

where $\beta_g(\frac{1}{2}) = \frac{P(1)}{\zeta_{\mathbb{A}}(2)}(g+1 + \zeta_{\mathbb{A}}(\frac{1}{2}) + \frac{2}{\log q} \frac{P'}{P}(1))$, $\beta_g(s) = \frac{\zeta_{\mathbb{A}}(2s)}{\zeta_{\mathbb{A}}(2)}P(2s)$ for $\operatorname{Re}(s) \geq 1$, and, for $\operatorname{Re}(s) < 1$ ($s \neq \frac{1}{2}$),

$$\beta_g(s) = \frac{\zeta_{\mathbb{A}}(2s)}{\zeta_{\mathbb{A}}(2)} \left\{ P(2s) - P(1)q^{(1-2s)(\lfloor \frac{g}{2} \rfloor + 1)} \right. \\ \left. + \frac{\zeta_{\mathbb{A}}(2-s)}{\zeta_{\mathbb{A}}(1+s)}q^{(1-2s)(g+1)} \left(P(1)q^{(2s-1)(\lfloor \frac{g-1}{2} \rfloor + 1)} - P(2-2s) \right) \right\} \\ - P(1)q^{-gs} \left(q^{\lfloor \frac{g}{2} \rfloor - s} + \frac{\zeta_{\mathbb{A}}(2-s)}{\zeta_{\mathbb{A}}(1+s)}q^{\lfloor \frac{g-1}{2} \rfloor} \right).$$

(3) For an arbitrary small $\varepsilon > 0$ and $s \in \bar{X}_\varepsilon$, we have

$$\sum_{u \in \mathcal{F}'_{g+1}} L(s, \chi_u) = \gamma_g(s)q^{2g+2} + \begin{cases} O(2^{\frac{g}{2}}q^{(2-s)g}) & \text{if } \operatorname{Re}(s) < s_1, \\ O(gq^{\frac{3g}{2}}) & \text{if } \operatorname{Re}(s) \geq s_1, \end{cases}$$

where $\gamma_g(\frac{1}{2}) = \frac{P(1)}{\zeta_{\mathbb{A}}(2)}(g+1 + \frac{\zeta_{\mathbb{A}}(0)}{\zeta_{\mathbb{A}}(\frac{1}{2})} + \frac{2}{\log q} \frac{P'}{P}(1))$, $\gamma_g(s) = \frac{\zeta_{\mathbb{A}}(2s)}{\zeta_{\mathbb{A}}(2)}P(2s)$ for $\operatorname{Re}(s) \geq 1$, and, for $s \in X_\varepsilon$,

$$\gamma_g(s) = \frac{\zeta_{\mathbb{A}}(2s)}{\zeta_{\mathbb{A}}(2)} \left\{ P(2s) - P(1)q^{(1-2s)(\lfloor \frac{g}{2} \rfloor + 1)} \right. \\ \left. + \left(\frac{1+q^{-s}}{1+q^{s-1}} \right) q^{(1-2s)(g+1)} \left(P(1)q^{(2s-1)(\lfloor \frac{g-1}{2} \rfloor + 1)} - P(2-2s) \right) \right\} \\ + (-1)^g P(1)q^{-gs} \left(q^{\lfloor \frac{g}{2} \rfloor - s} - \left(\frac{1+q^{-s}}{1+q^{s-1}} \right) q^{\lfloor \frac{g-1}{2} \rfloor} \right).$$

Remark 2.5. The restrictions of $|s-\frac{1}{2}| > \varepsilon$ in X_ε and $|s-1| > \delta$ in Theorem 2.4 are caused by the facts that $\zeta_{\mathbb{A}}(2s)$ and $\zeta_{\mathbb{A}}(2-s)$ are unbounded as $s \rightarrow \frac{1}{2}$ and $s \rightarrow 1$, respectively.

Since $\#\mathcal{I}_{g+1} = 2\zeta_{\mathbb{A}}(2)^{-1}q^{2g+1}$ and $\#\mathcal{F}_{g+1} = \#\mathcal{F}'_{g+1} = \zeta_{\mathbb{A}}(2)^{-1}q^{2g+2}$, we get from Theorem 2.4 the following asymptotic mean value formulas of L -functions at $s = \frac{1}{2}$ and $s = 1$. Compare with the results in [1, 2, 12, 13].

Corollary 2.6. Assume that $q > 2$ is fixed. As $g \rightarrow \infty$, we have

$$(1) \frac{1}{\#\mathcal{I}_{g+1}} \sum_{u \in \mathcal{I}_{g+1}} L(\frac{1}{2}, \chi_u) \sim (g+1)P(1),$$

$$(2) \frac{1}{\#\mathcal{F}_{g+1}} \sum_{u \in \mathcal{F}_{g+1}} L(\frac{1}{2}, \chi_u) \sim (g+1)P(1),$$

$$(3) \frac{1}{\#\mathcal{F}'_{g+1}} \sum_{u \in \mathcal{F}'_{g+1}} L(\tfrac{1}{2}, \chi_u) \sim (g+1)P(1).$$

Corollary 2.7. *Assume that $q > 2$ is fixed. As $g \rightarrow \infty$, we have*

$$\begin{aligned} (1) \quad & \frac{1}{\#\mathcal{I}_{g+1}} \sum_{u \in \mathcal{I}_{g+1}} L(1, \chi_u) \sim \zeta_{\mathbb{A}}(2)P(2), \\ (2) \quad & \frac{1}{\#\mathcal{F}_{g+1}} \sum_{u \in \mathcal{F}_{g+1}} L(1, \chi_u) \sim \zeta_{\mathbb{A}}(2)P(2), \\ (3) \quad & \frac{1}{\#\mathcal{F}'_{g+1}} \sum_{u \in \mathcal{F}'_{g+1}} L(1, \chi_u) \sim \zeta_{\mathbb{A}}(2)P(2). \end{aligned}$$

Let \mathcal{O}_u be the integral closure of \mathbb{A} in the quadratic function field K_u and h_u be the ideal class number of \mathcal{O}_u . If K_u is real, R_u denotes the regulator of \mathcal{O}_u . We have the following formula which connects $L(1, \chi_u)$ with h_u ([6, Theorem 5.2]):

$$L(1, \chi_u) = \begin{cases} q^{-g_u} h_u & \text{if } K_u \text{ is ramified imaginary,} \\ \zeta_{\mathbb{A}}(2)^{-1} q^{-g_u} h_u R_u & \text{if } K_u \text{ is real,} \\ \frac{1}{2} \zeta_{\mathbb{A}}(2) \zeta_{\mathbb{A}}(3)^{-1} q^{-g_u} h_u & \text{if } K_u \text{ is inert imaginary.} \end{cases} \quad (2.8)$$

From Theorem 2.4 and (2.8), we obtain the following theorem.

Theorem 2.8. *As q is fixed and $g \rightarrow \infty$, we have*

$$\begin{aligned} (1) \quad & \sum_{u \in \mathcal{I}_{g+1}} h_u = 2P(2)q^{3g+1} + O\left(gq^{\frac{5}{2}g}\right), \\ (2) \quad & \sum_{u \in \mathcal{F}_{g+1}} h_u R_u = \zeta_{\mathbb{A}}(2)P(2)q^{3g+2} + O\left(gq^{\frac{5}{2}g}\right), \\ (3) \quad & \sum_{u \in \mathcal{F}'_{g+1}} h_u = 2\zeta_{\mathbb{A}}(2)^{-1} \zeta_{\mathbb{A}}(3)P(2)q^{3g+2} + O\left(gq^{\frac{5}{2}g}\right). \end{aligned}$$

Corollary 2.9. *As q is fixed and $g \rightarrow \infty$, we have*

$$\begin{aligned} (1) \quad & \frac{1}{\#\mathcal{I}_{g+1}} \sum_{u \in \mathcal{I}_{g+1}} h_u \sim \zeta_{\mathbb{A}}(2)P(2)q^g, \\ (2) \quad & \frac{1}{\#\mathcal{F}_{g+1}} \sum_{u \in \mathcal{F}_{g+1}} h_u R_u \sim \zeta_{\mathbb{A}}(2)^2 P(2)q^g, \\ (3) \quad & \frac{1}{\#\mathcal{F}'_{g+1}} \sum_{u \in \mathcal{F}'_{g+1}} h_u \sim 2\zeta_{\mathbb{A}}(3)P(2)q^g. \end{aligned}$$

3. PRELIMINARIES

3.1. “Approximate” functional equations of L -functions. Let $u \in k$ be a normalized one as in (2.3) and write

$$L(s, \chi_u) = \sum_{n=0}^{d_u} A_u(n) q^{-sn} \quad \text{with } A_u(n) = \sum_{f \in \mathbb{A}_n^+} \chi_u(f),$$

where $d_u = 2g_u + \frac{1}{2}(1 + (-1)^{\varepsilon(u)})$. In this subsection we prove the following lemma, which is a generalization of Lemmas 2.1 in [1, 12, 13]. We remark that the proof of Lemma 3.1 also can be applied to obtain “Approximate” functional equations of L -functions $L(s, \chi_N)$ in odd characteristic.

Lemma 3.1. *Let $s \in \mathbb{C}$ with $\operatorname{Re}(s) \geq \frac{1}{2}$.*

(1) For $u \in \mathcal{I}$, we have

$$L(s, \chi_u) = \sum_{n=0}^{g_u} A_u(n) q^{-sn} + q^{(1-2s)g_u} \sum_{n=0}^{g_u-1} A_u(n) q^{(s-1)n}. \quad (3.1)$$

(2) For $u \in \mathcal{F}$, we have

$$L(s, \chi_u) = \sum_{n=0}^{g_u} A_u(n) q^{-sn} - q^{-(g_u+1)s} \sum_{n=0}^{g_u} A_u(n) + H_u(s), \quad (3.2)$$

where $H_u(1) := \zeta_{\mathbb{A}}(2)^{-1} q^{-g_u} \sum_{n=0}^{g_u-1} (g_u - n) A_u(n)$ and, for $s \neq 1$,

$$H_u(s) := q^{(1-2s)g_u} \eta(s) \sum_{n=0}^{g_u-1} q^{(s-1)n} A_u(n) - q^{-sg_u} \eta(s) \sum_{n=0}^{g_u-1} A_u(n)$$

with $\eta(s) = \frac{\zeta_{\mathbb{A}}(2-s)}{\zeta_{\mathbb{A}}(1+s)}$.

(3) For $u \in \mathcal{F}'$, we have

$$\begin{aligned} L(s, \chi_u) &= \sum_{n=0}^{g_u} A_u(n) q^{-sn} + q^{-(g_u+1)s} \sum_{n=0}^{g_u} (-1)^{n+g_u} A_u(n) \\ &\quad + \nu(s) q^{(1-2s)g_u} \sum_{n=0}^{g_u-1} A_u(n) q^{(s-1)n} + \nu(s) q^{-sg_u} \sum_{n=0}^{g_u-1} (-1)^{n+g_u+1} A_u(n) \end{aligned} \quad (3.3)$$

with $\nu(s) = \frac{1+q^{-s}}{1+q^{s-1}}$.

Proof. Write

$$\mathcal{L}(z, \chi_u) = \sum_{n=0}^{d_u} A_u(n) z^n \quad \text{and} \quad \mathcal{L}^*(z, \chi_u) = \sum_{n=0}^{2g_u} A_u^*(n) z^n.$$

By definition (2.5), we have

$$A_u^*(n) = \begin{cases} A_u(n) & \text{if } u \in \mathcal{I}, \\ \sum_{i=0}^n A_u(i) & \text{if } u \in \mathcal{F}, \\ \sum_{i=0}^n (-1)^{n-i} A_u(i) & \text{if } u \in \mathcal{F}'. \end{cases} \quad (3.4)$$

From the functional equation (2.6), we have

$$\sum_{n=0}^{2g_u} A_u^*(n) z^n = \sum_{n=0}^{2g_u} A_u^*(n) q^{g_u-n} z^{2g_u-n} = \sum_{n=0}^{2g_u} A_u^*(2g_u-n) q^{n-g_u} z^n,$$

and equating coefficients, we have

$$A_u^*(n) = A_u^*(2g_u-n) q^{n-g_u} \quad \text{or} \quad A_u^*(2g_u-n) = A_u^*(n) q^{g_u-n}.$$

Hence, we can write $\mathcal{L}^*(z, \chi_u)$ as

$$\mathcal{L}^*(z, \chi_u) = \sum_{n=0}^{g_u} A_u^*(n) z^n + q^{g_u} z^{2g_u} \sum_{n=0}^{g_u-1} A_u^*(n) q^{-n} z^{-n}. \quad (3.5)$$

If $u \in \mathcal{I}$, since $\mathcal{L}(u, \chi_u) = \mathcal{L}^*(u, \chi_u)$, (3.1) follows from (3.5) immediately by letting $z = q^{-s}$. Suppose that $u \in \mathcal{F}$. From (3.4) and (3.5), we have

$$\begin{aligned} \mathcal{L}^*(z, \chi_u) &= \sum_{n=0}^{g_u} \left(\sum_{i=0}^n A_u(i) \right) z^n + q^{g_u} z^{2g_u} \sum_{n=0}^{g_u-1} \left(\sum_{j=0}^n A_u(j) \right) q^{-n} z^{-n} \\ &= \sum_{n=0}^{g_u} \left(\frac{z^n - z^{g_u+1}}{1-z} \right) A_u(n) + H^*(z), \end{aligned} \quad (3.6)$$

where $H^*(q^{-1}) := q^{-g_u} \sum_{n=0}^{g_u-1} (g_u - n) A_u(n)$ and, for $z \neq q^{-1}$,

$$H^*(z) := \frac{q^{g_u} z^{2g_u}}{1 - q^{-1} z^{-1}} \sum_{n=0}^{g_u-1} q^{-n} z^{-n} A_u(n) - \frac{z^{g_u}}{1 - q^{-1} z^{-1}} \sum_{n=0}^{g_u-1} A_u(n).$$

Then, by multiplying $(1-z)$ on (3.6) and putting $z = q^{-s}$, we get (3.2). Finally, consider the case that $u \in \mathcal{F}'$. By (3.4) and (3.5), we have

$$\begin{aligned} \mathcal{L}^*(z, \chi_u) &= \sum_{n=0}^{g_u} \left(\sum_{i=0}^n (-1)^{n-i} A_u(i) \right) z^n + q^{g_u} z^{2g_u} \sum_{n=0}^{g_u-1} \left(\sum_{j=0}^n (-1)^{n-j} A_u(j) \right) q^{-n} z^{-n} \\ &= \frac{1}{1+z} \sum_{n=0}^{g_u} A_u(n) z^n + \frac{z^{g_u+1}}{1+z} \sum_{n=0}^{g_u} (-1)^{n+g_u} A_u(n) \\ &\quad + \frac{q^{g_u} z^{2g_u}}{1 + q^{-1} z^{-1}} \sum_{n=0}^{g_u-1} A_u(n) q^{-n} z^{-n} + \frac{z^{g_u}}{1 + q^{-1} z^{-1}} \sum_{n=0}^{g_u-1} (-1)^{n+g_u+1} A_u(n). \end{aligned} \quad (3.7)$$

By multiplying $(1+z)$ on (3.7) and putting $z = q^{-s}$, we get (3.3). \square

3.2. Some auxiliary lemmas. All results in this subsection hold in arbitrary characteristic. Thus we assume that q is a power of any prime number.

The following lemma is a minor modification of Theorem 17.1 in [15].

Lemma 3.2. *Let $\rho : \mathbb{A}^+ \rightarrow \mathbb{C}$ and let $\zeta_\rho(s)$ be the corresponding Dirichlet series. Suppose this series converges absolutely in the region $\operatorname{Re}(s) > 1$ and is holomorphic in the region $\{s \in B : \operatorname{Re}(s) = 1\}$ except for a simple pole of at $s = 1$ with residue α , where*

$$B = \left\{ s \in \mathbb{C} : -\frac{\pi i}{\log q} \leq \operatorname{Im}(s) < \frac{\pi i}{\log q} \right\}.$$

Then, there is a positive real number $\delta < 1$ such that

$$\sum_{f \in \mathbb{A}_n^+} \rho(f) = \alpha(\log q) q^n + O(q^{\delta n}).$$

If $\zeta_\rho(s) - \frac{\alpha}{s-1}$ is holomorphic in $\operatorname{Re}(s) \geq \delta'$, then the error term can be replaced by with $O(q^{\delta' n})$.

Lemma 3.3. *Let $L \in \mathbb{A}^+$. Given any $\epsilon > 0$, we have*

$$\sum_{\substack{f \in \mathbb{A}_n^+ \\ (f, L)=1}} \Phi(f) = \frac{1}{\zeta_{\mathbb{A}}(2)} q^{2n} \prod_{P|L} (1 + |P|^{-1})^{-1} + O(q^{(1+\epsilon)n}).$$

Proof. Let $\zeta_\Phi(s)$ be the Dirichlet series associated to Φ . It is known that ([15, Chap 2. equation (6)])

$$\zeta_\Phi(s) = \frac{\zeta_{\mathbb{A}}(s-1)}{\zeta_{\mathbb{A}}(s)}. \quad (3.8)$$

Let $\rho : \mathbb{A}^+ \rightarrow \mathbb{C}$ be defined by $\rho(f) = \Phi(f)$ if $(L, f) = 1$ and $\rho(f) = 0$ otherwise, and $\zeta_\rho(s)$ be the Dirichlet series associated to ρ :

$$\zeta_\rho(s) := \sum_{\substack{f \in \mathbb{A}^+ \\ (f, L) = 1}} \Phi(f) |f|^{-s} = \prod_{\substack{P \in \mathbb{P} \\ P \nmid L}} \left(1 + \sum_{n=1}^{\infty} \Phi(P^n) |P|^{-ns} \right). \quad (3.9)$$

Then, we have

$$\zeta_\rho(s) = \zeta_\Phi(s) \prod_{\substack{P \in \mathbb{P} \\ P \mid L}} \left(\frac{1 - |P|^{1-s}}{1 - |P|^{-s}} \right),$$

which has a simple pole at $s = 2$ and is holomorphic in the region $\operatorname{Re}(s) > 1$. Hence, $\zeta_\rho(s+1)$ has a simple pole at $s = 1$ and is holomorphic in the region $\operatorname{Re}(s) > 0$. Then $\zeta_\rho(s+1)$ is holomorphic in the region $\operatorname{Re}(s) \geq \epsilon$ except for a simple pole at $s = 1$ with residue

$$\alpha = \frac{1}{\zeta_{\mathbb{A}}(2) \log q} \prod_{P \mid L} (1 + |P|^{-1})^{-1}.$$

Applying Lemma 3.2 to $\zeta_\rho(s+1)$ with $\delta' = \epsilon$, we find

$$\sum_{\substack{f \in \mathbb{A}_n^+ \\ (f, L) = 1}} \Phi(f) = \frac{1}{\zeta_{\mathbb{A}}(2)} q^{2n} \prod_{P \mid L} (1 + |P|^{-1})^{-1} + O\left(q^{(1+\epsilon)n}\right).$$

□

Applying Lemma 3.3 with $\epsilon = \frac{1}{2}$, we have the following corollary.

Corollary 3.4. *We have*

$$\sum_{L \in \mathbb{A}_l^+} \sum_{\substack{f \in \mathbb{A}_n^+ \\ (f, L) = 1}} \Phi(f) = \frac{1}{\zeta_{\mathbb{A}}(2)} q^{2n+l} \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq l}} \frac{\mu(D)}{|D|} \prod_{P \mid D} \frac{1}{|P|+1} + O\left(q^{\frac{3n}{2}+l}\right). \quad (3.10)$$

Proof. By Lemma 3.3, we have

$$\sum_{L \in \mathbb{A}_l^+} \sum_{\substack{f \in \mathbb{A}_n^+ \\ (f, L) = 1}} \Phi(f) = \frac{1}{\zeta_{\mathbb{A}}(2)} q^{2n} \sum_{L \in \mathbb{A}_l^+} \prod_{P \mid L} (1 + |P|^{-1})^{-1} + O\left(q^{\frac{3n}{2}+l}\right),$$

and, by [2, Lemma 5.7],

$$\sum_{L \in \mathbb{A}_l^+} \prod_{P \mid L} (1 + |P|^{-1})^{-1} = q^l \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq l}} \frac{\mu(D)}{|D|} \prod_{P \mid D} \frac{1}{|P|+1}.$$

So we get the result. □

Now we have the following lemma, which is a generalization of Lemmas 3.3, 3.4 and 3.5 of [13].

Lemma 3.5. *Let $h \in \{g-1, g\}$. For any $s \in \mathbb{C}$ with $\operatorname{Re}(s) \geq \frac{1}{2}$, we have*

(1)

$$\sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq [\frac{h}{2}]}} \frac{\mu(D)}{|D|^{2s}} \prod_{P|D} \frac{1}{|P|+1} = P(2s) + O(q^{-sg}), \quad (3.11)$$

(2)

$$\sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq [\frac{h}{2}]}} \frac{\mu(D)}{|D|^{2-2s}} \prod_{P|D} \frac{1}{|P|+1} = \begin{cases} P(2-2s) + O(q^{(s-1)g}) & \text{if } \operatorname{Re}(s) < 1, \\ O(g) & \text{if } s = 1, \\ O(q^{(s-1)g}) & \text{if } \operatorname{Re}(s) \geq 1 \ (s \neq 1). \end{cases} \quad (3.12)$$

Proof. (1) We can write

$$\begin{aligned} \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq [\frac{h}{2}]}} \frac{\mu(D)}{|D|^{2s}} \prod_{P|D} \frac{1}{|P|+1} &= \sum_{D \in \mathbb{A}^+} \frac{\mu(D)}{|D|^{2s}} \prod_{P|D} \frac{1}{|P|+1} \\ &\quad - \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) > [\frac{h}{2}]}} \frac{\mu(D)}{|D|^{2s}} \prod_{P|D} \frac{1}{|P|+1}. \end{aligned} \quad (3.13)$$

Using the Euler product formula, we can show that

$$\sum_{D \in \mathbb{A}^+} \frac{\mu(D)}{|D|^s} \prod_{P|D} \frac{1}{|P|+1} = \prod_P \left(1 - \frac{1}{|P|^s(|P|+1)} \right) = P(s).$$

Hence, we have

$$\sum_{D \in \mathbb{A}^+} \frac{\mu(D)}{|D|^{2s}} \prod_{P|D} \frac{1}{|P|+1} = P(2s). \quad (3.14)$$

We also have

$$\begin{aligned} \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) > [\frac{h}{2}]}} \frac{\mu(D)}{|D|^{2s}} \prod_{P|D} \frac{1}{|P|+1} &\ll \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) > [\frac{h}{2}]}} \frac{\mu(D)^2}{|D|^{2s}} \prod_{P|D} \frac{1}{|P|} \ll \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) > [\frac{h}{2}]}} |D|^{-(2s+1)} \\ &= \sum_{n > [\frac{h}{2}]} \sum_{D \in \mathbb{A}_n^+} q^{-(2s+1)n} = \sum_{n > [\frac{h}{2}]} q^{-2sn} \ll q^{-sg}. \end{aligned} \quad (3.15)$$

By inserting (3.14) and (3.15) into (3.13), we get (3.11).

(2) If $\operatorname{Re}(s) < 1$, by a similar process as in the proof of (1), we can get

$$\sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq [\frac{h}{2}]}} \frac{\mu(D)}{|D|^{2-2s}} \prod_{P|D} \frac{1}{|P|+1} = P(2-2s) + O(q^{(s-1)g}).$$

If $s = 1$, we have

$$\sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq [\frac{h}{2}]}} \mu(D) \prod_{P|D} \frac{1}{|P|+1} \leq \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq [\frac{h}{2}]}} \mu^2(D) \prod_{P|D} \frac{1}{|P|} \leq \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq [\frac{h}{2}]}} |D|^{-1} \leq g.$$

Finally, if $\operatorname{Re}(s) \geq 1$ ($s \neq 1$), we have

$$\begin{aligned} \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq [\frac{h}{2}]}} \frac{\mu(D)}{|D|^{2-2s}} \prod_{P|D} \frac{1}{|P|+1} &\ll \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq [\frac{h}{2}]}} \mu^2(D) |D|^{2s-2} \prod_{P|D} \frac{1}{|P|} \\ &\ll \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq [\frac{h}{2}]}} |D|^{2s-3} = \sum_{n=0}^{[\frac{h}{2}]} q^{(2s-2)n} \ll q^{(s-1)g}. \end{aligned}$$

□

Recall that, for a small $\varepsilon > 0$,

$$\bar{X}_\varepsilon = \{\tfrac{1}{2}\} \cup X_\varepsilon \cup \{s \in \mathbb{C} : \operatorname{Re}(s) \geq 1\},$$

where $X_\varepsilon = \{s \in \mathbb{C} : \tfrac{1}{2} \leq \operatorname{Re}(s) < 1, |s - \tfrac{1}{2}| > \varepsilon\}$. For $h \in \{g-1, g\}$ and $s \in \bar{X}_\varepsilon$, consider the following two summations

$$J_h(s) = \sum_{l=0}^{[\frac{h}{2}]} q^{(1-2s)l} \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq l}} \frac{\mu(D)}{|D|} \prod_{P|D} \frac{1}{|P|+1} \quad (3.16)$$

and

$$\tilde{J}_h(s) = \sum_{l=0}^{[\frac{h}{2}]} q^{(2s-1)l} \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq l}} \frac{\mu(D)}{|D|} \prod_{P|D} \frac{1}{|P|+1}, \quad (3.17)$$

which will be used repeatedly in §4.1. Recall that (see (5.29) in [2])

$$P'(s) = P(s) \log q \sum_{P \in \mathbb{P}} \frac{\deg(P)}{|P|^s(|P|+1) - 1}. \quad (3.18)$$

Lemma 3.6. *Let $h \in \{g-1, g\}$. For a small $\varepsilon > 0$ and $s \in X_\varepsilon \cup \{s \in \mathbb{C} : \operatorname{Re}(s) \geq 1\}$, we have*

$$J_h(s) = \zeta_{\mathbb{A}}(2s) \left(P(2s) - q^{(1-2s)([\frac{h}{2}]+1)} P(1) \right) + O(q^{-sg}), \quad (3.19)$$

and for $s = \frac{1}{2}$, we have

$$J_h(\tfrac{1}{2}) = \left([\tfrac{h}{2}] + 1 + \frac{1}{\log q} \frac{P'}{P}(1) \right) P(1) + O\left(gq^{-\frac{g}{2}}\right). \quad (3.20)$$

Proof. By change of summations, we have

$$J_h(s) = \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq [\frac{h}{2}]}} \frac{\mu(D)}{|D|} \prod_{P|D} \frac{1}{|P|+1} \sum_{\deg(D) \leq m \leq [\frac{h}{2}]} q^{(1-2s)m}.$$

If $s = \frac{1}{2}$, by (3.11) and [2, Lemma 5.11, Proposition 5.12], we have

$$\begin{aligned} J_h(\tfrac{1}{2}) &= ([\tfrac{h}{2}] + 1) \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq [\frac{h}{2}]}} \frac{\mu(D)}{|D|} \prod_{P|D} \frac{1}{|P|+1} - \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq [\frac{h}{2}]}} \frac{\mu(D) \deg(D)}{|D|} \prod_{P|D} \frac{1}{|P|+1} \\ &= \left([\tfrac{h}{2}] + 1 + \frac{1}{\log q} \frac{P'}{P}(1) \right) P(1) + O\left(gq^{-\frac{g}{2}}\right). \end{aligned}$$

For $s \in X_\varepsilon \cup \{s \in \mathbb{C} : \operatorname{Re}(s) \geq 1\}$, $\zeta_{\mathbb{A}}(2s)$ is bounded. Now replacing $\sum_{\deg(D) \leq m \leq [\frac{h}{2}]} q^{(1-2s)m}$ by

$$\frac{q^{(1-2s)\deg(D)} - q^{(1-2s)([\frac{h}{2}]+1)}}{1 - q^{1-2s}} = \zeta_{\mathbb{A}}(2s) \left(q^{(1-2s)\deg(D)} - q^{(1-2s)([\frac{h}{2}]+1)} \right),$$

we have, by Lemma 3.5 (1),

$$\begin{aligned} J_h(s) &= \zeta_{\mathbb{A}}(2s) \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq [\frac{h}{2}]}} \frac{\mu(D)}{|D|^{2s}} \prod_{P|D} \frac{1}{|P|+1} \\ &\quad - \zeta_{\mathbb{A}}(2s) q^{(1-2s)([\frac{h}{2}]+1)} \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq [\frac{h}{2}]}} \frac{\mu(D)}{|D|} \prod_{P|D} \frac{1}{|P|+1} \\ &= \zeta_{\mathbb{A}}(2s) \left(P(2s) - q^{(1-2s)([\frac{h}{2}]+1)} P(1) \right) + O(q^{-sg}). \end{aligned}$$

□

Lemma 3.7. *Let $h \in \{g-1, g\}$. For a small $\varepsilon > 0$ and $s \in \bar{X}_\varepsilon$, we have*

$$\tilde{J}_h(s) = \begin{cases} ([\frac{h}{2}] + 1 + \frac{1}{\log q} \frac{P'}{P}(1)) P(1) + O(gq^{-\frac{g}{2}}) & \text{if } s = \frac{1}{2}, \\ \zeta_{\mathbb{A}}(2s) (q^{(2s-1)[\frac{h}{2}]} P(1) - q^{(1-2s)} P(2-2s)) + O(q^{(s-1)g}) & \text{if } s \in X_\varepsilon, \\ \zeta_{\mathbb{A}}(2s) q^{(2s-1)[\frac{h}{2}]} P(1) + O(gq^{(s-1)g}) & \text{if } \operatorname{Re}(s) \geq 1. \end{cases}$$

Proof. We can write

$$\tilde{J}_h(s) = \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq [\frac{h}{2}]}} \frac{\mu(D)}{|D|} \prod_{P|D} \frac{1}{|P|+1} \sum_{\deg(D) \leq m \leq [\frac{h}{2}]} q^{(2s-1)m}.$$

If $s = \frac{1}{2}$, by (3.11) and [2, Lemma 5.11, Proposition 5.12], we have

$$\begin{aligned} \tilde{J}_h(\frac{1}{2}) &= ([\frac{h}{2}] + 1) \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq [\frac{h}{2}]}} \frac{\mu(D)}{|D|} \prod_{P|D} \frac{1}{|P|+1} - \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq [\frac{h}{2}]}} \frac{\mu(D) \deg(D)}{|D|} \prod_{P|D} \frac{1}{|P|+1} \\ &= \left([\frac{h}{2}] + 1 + \frac{1}{\log q} \frac{P'}{P}(1) \right) P(1) + O\left(gq^{-\frac{g}{2}}\right). \end{aligned}$$

For $s \neq \frac{1}{2}$, as in the proof of Lemma 3.6, we have

$$\begin{aligned} \tilde{J}_h(s) &= \zeta_{\mathbb{A}}(2s) q^{(2s-1)[\frac{h}{2}]} \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq [\frac{h}{2}]}} \frac{\mu(D)}{|D|} \prod_{P|D} \frac{1}{|P|+1} \\ &\quad - \zeta_{\mathbb{A}}(2s) q^{(1-2s)} \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq [\frac{h}{2}]}} \frac{\mu(D)}{|D|^{2-2s}} \prod_{P|D} \frac{1}{|P|+1}. \end{aligned}$$

Now, the results follows from (3.11) and (3.12). □

3.3. Two sums. For each $M \in \mathcal{B}$, let \mathcal{C}_M be the set of rational functions $u \in \mathcal{C}$ whose denominator divides M , $\mathcal{E}_M = \mathcal{E} \cap \mathcal{C}_M$ and $\mathcal{F}_M = \mathcal{F} \cap \mathcal{C}_M$. Note that \mathcal{C}_M and \mathcal{E}_M are abelian groups under addition and $\#\mathcal{E}_M = |\tilde{M}|$, $\#\mathcal{F}_M = \Phi(\tilde{M})$. In fact, \mathcal{C}_M can be defined for any monic polynomial M , not necessarily in \mathcal{B} . Note that \mathcal{E}_n (resp. \mathcal{F}_n) is the disjoint union of \mathcal{E}_M (resp. \mathcal{F}_M) with $M \in \mathcal{B}_n$.

For any $f \in \mathbb{A}^+$ and $M \in \mathcal{B}$, we define

$$U_{f,M} := \sum_{u \in \mathcal{C}_M} \left\{ \frac{u}{f} \right\}, \quad \Gamma_{f,M} := \sum_{u \in \mathcal{E}_M} \left\{ \frac{u}{f} \right\} \quad \text{and} \quad T_{f,M} := \sum_{u \in \mathcal{F}_M} \left\{ \frac{u}{f} \right\}.$$

Lemma 3.8. *Let $f \in \mathbb{A}^+$ and $M \in \mathcal{B}$. If $\deg(f) \leq \deg(t(M))$, $\gcd(M, f) = 1$ and f is not a perfect square, then we have $U_{f,M} = \Gamma_{f,M} = 0$.*

Proof. It is easy to see that if $\deg(f) \leq \deg(t(M))$, then $\mathcal{C}_{t(M)}$ contains a complete system modulo f . Note that every element in \mathcal{E}_M can be obtained by normalization from an element of $\mathcal{C}_{t(M)}$ and the normalization process preserves the value $\{ \frac{u}{f} \}$. Then, since $\mathcal{C}_{t(M)}$ and \mathcal{E}_M are abelian groups, we get the result. \square

Note that if $\gcd(M, f) = 1$ and there is a $u \in \mathcal{E}_M$ with $\{ \frac{u}{f} \} = -1$, then $\Gamma_{f,M} = 0$. Thus, we have $\Gamma_{f,M} = 0$ or $|\tilde{M}| = q^{\deg(t(M))/2}$.

We have

$$\Gamma_{f,M} = \sum_{\substack{N \in \mathcal{B} \\ N|M}} T_{f,N}$$

and thus

$$T_{f,M} = \sum_{N|\tilde{M}} \mu(N) \Gamma_{f,[M/N]},$$

where $[M/N] = (\tilde{M}/N)^*$ for $N|\tilde{M}$. Recall that $\tilde{M} := \sqrt{t(M)}$ and $M^* := M^2/r(M)$.

Lemma 3.9. *Let $M, N \in \mathcal{B}$ with $N|M$. If $\Gamma_{f,N} = 0$, then $\Gamma_{f,M} = 0$. Thus, if $\Gamma_{f,M} \neq 0$, then $\Gamma_{f,N} \neq 0$.*

Lemma 3.10. *Let $M, N \in \mathcal{B}$. If $\gcd(M, N) = 1$, then $\Gamma_{f,MN} = \Gamma_{f,M} \Gamma_{f,N}$. Thus, for $M = P_1^{e_1} \cdots P_r^{e_r} \in \mathcal{B}$, we have $\Gamma_{f,M} = \Gamma_{f,P_1^{e_1}} \cdots \Gamma_{f,P_r^{e_r}}$.*

Proof. This follows from the fact that $u \in \mathcal{E}_{MN}$ can be uniquely written as $u_1 + u_2$ with $u_1 \in \mathcal{E}_M$ and $u_2 \in \mathcal{E}_N$ and the fact that $\{ \frac{u_1+u_2}{f} \} = \{ \frac{u_1}{f} \} \{ \frac{u_2}{f} \}$. \square

Corollary 3.11. *For each positive integer m and a non-square $f \in \mathbb{A}_d^+$, there exist at most $\binom{[d/2]}{m}$ polynomials $M \in \mathbb{A}_m^+$ with $\Gamma_{f,M^*} \neq 0$ and $\gcd(M, f) = 1$.*

Proof. Let $M_1, \dots, M_s \in \mathbb{A}_m^+$ be distinct polynomials such that $\Gamma_{f,M_i^*} \neq 0$, $1 \leq i \leq s$. For $M := \text{lcm}(M_1, \dots, M_s)$, by Lemma 3.9 and Lemma 3.10, we have $\Gamma_{f,M^*} \neq 0$, and so $2\deg(M) < d$. Note that if $s > \binom{[d/2]}{m}$, then $\deg(M) > m + \lfloor \frac{d}{2} \rfloor - m$, which implies that $2\deg(M) \geq d$ and so $\Gamma_{f,M^*} = 0$. Thus we should have $s \leq \binom{[d/2]}{m}$ and the result follows. \square

Now the following proposition is immediate, which can be viewed as an analogue of [2, Lemma 6.2].

Proposition 3.12. *Let $f \in \mathbb{A}_d^+$ be non-square and m be a positive integer with $2m < d$. Then we have*

$$\sum_{u \in \mathcal{E}_m(f)} \left\{ \frac{u}{f} \right\} = \sum_{M \in \mathbb{A}_m^+} \Gamma_{f,M^*} \leq \binom{[d/2]}{m} q^m,$$

where $\mathcal{E}_m(f)$ is the set of elements $u \in \mathcal{E}_m$ such that $u \in \mathcal{E}_M$ for some $M \in \mathcal{B}_m$ with $\gcd(M, f) = 1$.

Lemma 3.13. *Let $M \in \mathcal{B}$ and $f \in \mathbb{A}^+$ with $\deg(f) \leq \deg(M)$.*

- (1) *If $\gcd(f, M) \neq 1$, then $T_{f,M} = 0$.*
- (2) *If $\gcd(f, M) = 1$ and f is a perfect square, then $T_{f,M} = \Phi(\tilde{M})$.*

(3) If $\gcd(f, M) = 1$ and f is not a perfect square, then

$$T_{f,M} = \sum_{\substack{N|\tilde{M} \\ \deg([M/N]) < \deg(f)}} \mu(N) \Gamma_{f,[M/N]}.$$

Note that if $u \in \mathcal{E}_{g+1}$, then the genus of the curve defined by $X^2 + X + u = 0$ is g .

Lemma 3.14. *Let n be a positive integer. For any $L \in \mathcal{B}_\ell$ with $\ell < n$, there are $(1-q^{-1})q^{n-\ell}$ square-free monic polynomials $N \in \mathbb{A}^+$ such that there exists $M \in \mathcal{B}_n$ with $[M/N] = L$.*

Proof. It follows easily from the fact that $t(M) = t([M/N])N^2$ if N is square-free with $N|\tilde{M}$. \square

Proposition 3.15. *Let n be a positive integer. For any monic polynomial $f \in \mathbb{A}_d^+$, which is not a perfect square, we have*

$$\sum_{u \in \mathcal{F}_n} \left\{ \frac{u}{f} \right\} \ll 2^{\frac{d}{2}} q^n.$$

Proof. Let $L \in \mathcal{B}_\ell$ with $2\ell < d$. For each square-free monic polynomial N of degree $n - \ell$, there exists a unique $M \in \mathcal{B}_n$ such that $L = [M/N]$ and vice versa. Then, by Lemmas 3.8 and 3.13, we have

$$\begin{aligned} \sum_{u \in \mathcal{F}_n} \left\{ \frac{u}{f} \right\} &= \sum_{\substack{M \in \mathcal{B}_n \\ (M,f)=1}} T_{f,M} = \sum_{\ell=0}^{\lfloor \frac{d}{2} \rfloor - 1} \sum_{\substack{L \in \mathcal{B}_\ell \\ (L,f)=1}} \sum_{\substack{N \in \mathbb{A}_{n-\ell}^+ \\ (N,f)=1}} \mu(N) \Gamma_{f,L} \\ &\leq \sum_{\ell=0}^{\lfloor \frac{d}{2} \rfloor - 1} (1 - q^{-1}) q^{n-\ell} \sum_{\substack{L \in \mathcal{B}_\ell \\ (L,f)=1}} \Gamma_{f,L} \\ &\leq \sum_{\ell=0}^{\lfloor \frac{d}{2} \rfloor - 1} (1 - q^{-1}) q^{n-\ell} \binom{\lfloor d/2 \rfloor}{\ell} q^\ell \ll 2^{\frac{d}{2}} q^n. \end{aligned}$$

\square

Now we consider the ramified imaginary case. Let s be a positive integer. For $M \in \mathcal{B}$ and $\alpha \in \mathbb{F}_q^\times$, we define (inside the field k)

$$\begin{aligned} \mathcal{C}'_{M,s} &:= \mathcal{C}_{t(M)} + \mathbb{A}_{\leq 2s-2}, & \mathcal{C}_{M,s,\alpha} &:= \mathcal{C}'_{M,s} + \alpha T^{2s-1}, \\ \mathcal{E}'_{M,s} &:= \mathcal{E}_M + \tilde{\mathcal{G}}_{s-1}, & \mathcal{E}_{M,s,\alpha} &:= \mathcal{E}'_{M,s} + \alpha T^{2s-1}, \\ \mathcal{F}'_{M,s} &:= \mathcal{F}_M + \tilde{\mathcal{G}}_{s-1}, & \mathcal{F}_{M,s,\alpha} &:= \mathcal{F}'_{M,s} + \alpha T^{2s-1}, \end{aligned}$$

where $\mathbb{A}_{\leq m} = \{f \in \mathbb{A} : \deg(f) \leq m\}$, $\tilde{\mathcal{G}}_0 = \{0\}$ and $\tilde{\mathcal{G}}_{s-1}$ denotes the set of all polynomials $F(T) \in \mathbb{A}_{\leq 2s-1}$ of the form $F(T) = \sum_{i=1}^{s-1} \alpha_i T^{2i-1}$ for $s \geq 2$. For $f \in \mathbb{A}^+$, we also define

$$\begin{aligned} \tilde{U}'_{f,M,s} &:= \sum_{u \in \mathcal{C}'_{M,s}} \left\{ \frac{u}{f} \right\}, & \tilde{\Gamma}'_{f,M,s} &:= \sum_{u \in \mathcal{E}'_{M,s}} \left\{ \frac{u}{f} \right\}, & \tilde{T}'_{f,M,s} &:= \sum_{u \in \mathcal{F}'_{M,s}} \left\{ \frac{u}{f} \right\}, \\ \tilde{U}_{f,M,s,\alpha} &:= \sum_{u \in \mathcal{C}_{M,s,\alpha}} \left\{ \frac{u}{f} \right\}, & \tilde{\Gamma}_{f,M,s,\alpha} &:= \sum_{u \in \mathcal{E}_{M,s,\alpha}} \left\{ \frac{u}{f} \right\}, & \tilde{T}_{f,M,s,\alpha} &:= \sum_{u \in \mathcal{F}_{M,s,\alpha}} \left\{ \frac{u}{f} \right\}, \\ \tilde{U}_{f,M,s} &:= \sum_{\alpha \in \mathbb{F}_q^\times} \tilde{U}_{f,M,s,\alpha}, & \tilde{\Gamma}_{f,M,s} &:= \sum_{\alpha \in \mathbb{F}_q^\times} \tilde{\Gamma}_{f,M,s,\alpha}, & \tilde{T}_{f,M,s} &:= \sum_{\alpha \in \mathbb{F}_q^\times} \tilde{T}_{f,M,s,\alpha}. \end{aligned}$$

Lemma 3.16. *Let $r \geq 0$ and $s \geq 1$ be integers, $M \in \mathcal{B}_r$ and $\alpha \in \mathbb{F}_q^\times$. For any $f \in \mathbb{A}_m^+$ with $\gcd(M, f) = 1$ and $m \leq 2r + 2s - 2$, which is not a perfect square, we have*

$$\tilde{U}'_{f,M,s} = \tilde{U}_{f,M,s,\alpha} = \tilde{\Gamma}'_{f,M,s} = \tilde{\Gamma}_{f,M,s,\alpha} = 0.$$

Proof. If $\deg(A) < \deg(f) \leq 2r + 2s - 2$, then A can be expressed as $t(M)B + C$ with $\deg(C) < 2r$ and $\deg(B) = \deg(A) - 2r \leq 2s - 2$. Thus $\mathcal{C}'_{M,s}$ contains a complete residue system modulo f . Also, every element in $\mathcal{E}'_{M,s}$ can be obtained from $\mathcal{C}'_{M,s}$ by normalization. Since $\mathcal{C}'_{M,s}$ and $\mathcal{E}'_{M,s}$ are abelian groups, we have $\tilde{U}'_{f,M,s} = \tilde{\Gamma}'_{f,M,s} = 0$. Since $\tilde{U}_{f,M,s,\alpha} = \{\frac{\alpha T^{2s-1}}{f}\} \tilde{U}'_{f,M,s}$ and $\tilde{\Gamma}_{f,M,s,\alpha} = \{\frac{\alpha T^{2s-1}}{f}\} \tilde{\Gamma}'_{f,M,s}$, we also have $\tilde{U}_{f,M,s,\alpha} = \tilde{\Gamma}_{f,M,s,\alpha} = 0$. \square

As before we have

$$\tilde{\Gamma}'_{f,M,s} = \sum_{\substack{N \in \mathcal{B} \\ N|M}} \tilde{T}'_{f,N,s}$$

and thus

$$\tilde{T}'_{f,M,s} = \sum_{N|\tilde{M}} \mu(N) \tilde{\Gamma}'_{f,[M/N],s}.$$

Lemma 3.17. *Let $M, N \in \mathcal{B}$ with $N|M$. If $\tilde{\Gamma}'_{f,N,s} = 0$, then $\tilde{\Gamma}'_{f,M,s} = 0$.*

Lemma 3.18. *Let s be a positive integer, $M \in \mathcal{B}$ and $f \in A^+$ with $\deg(f) \leq \deg(t(M)) + 2s - 2$.*

- (1) *If $\gcd(f, M) \neq 1$, then $\tilde{T}'_{f,M,s} = 0$.*
- (2) *If $\gcd(f, M) = 1$ and f is a perfect square, then $\tilde{T}'_{f,M,s} = \Phi(\tilde{M})q^{s-1}$.*
- (3) *If $\gcd(f, M) = 1$ and f is not a perfect square, then*

$$\tilde{T}'_{f,M,s} = \sum_{\substack{N|\tilde{M} \\ \deg(t([M/N])) < \deg(f) - 2s + 2}} \mu(N) \tilde{\Gamma}'_{f,[M/N],s}.$$

For any positive integers r, s , let $\mathcal{E}_{(r,s)} = \bigcup_{M \in \mathcal{B}_r} \mathcal{E}'_{M,s}$. Note that if $\Gamma_{f,M} = 0$, then $\tilde{\Gamma}'_{f,M,s} = 0$ for any $s \geq 1$. Then, using similar process as in Proposition 3.12, we get

Lemma 3.19. *Let $r \geq 0$ and $s \geq 1$ be integers. For any $f \in \mathbb{A}_d^+$ with $d > 2r + 2s - 2$, which is not a perfect square, we have*

$$\sum_{u \in \mathcal{E}_{(r,s)}(f)} \left\{ \frac{u}{f} \right\} \leq \binom{[d/2] - s}{r} q^{r+s},$$

where $\mathcal{E}_{(r,s)}(f)$ is the set of elements $u \in \mathcal{E}_{(r,s)}$ such that $u \in \mathcal{E}'_{M,s}$ for some $M \in \mathcal{B}_r$ with $\gcd(M, f) = 1$.

Proposition 3.20. *Let $r \geq 0$ and $s \geq 1$ be integers. For any $f \in \mathbb{A}_d^+$ with $d \leq g = r + s - 1$, which is not a perfect square, we have*

$$\sum_{u \in \mathcal{I}_{(r,s)}} \left\{ \frac{u}{f} \right\} \leq 2^{\frac{d}{2}-s} q^{r+s}.$$

Thus we have

$$\sum_{u \in \mathcal{I}_{g+1}} \left\{ \frac{u}{f} \right\} \ll g 2^{\frac{d}{2}} q^g. \quad (3.21)$$

Proof. Write

$$\sum_{u \in \mathcal{I}_{(r,s)}} \left\{ \frac{u}{f} \right\} = \sum_{\alpha \in \mathbb{F}_q^*} \left\{ \frac{\alpha T^{2s-1}}{f} \right\} \left(\sum_{M \in \mathcal{B}_r} \tilde{T}'_{f,M,s} \right)$$

and, as in the proof of Proposition 3.15 using Lemmas 3.18 and 3.19, we have

$$\begin{aligned} \sum_{\substack{M \in \mathcal{B}_r \\ (M,f)=1}} \tilde{T}'_{f,M,s} &= \sum_{\ell=0}^{\lfloor \frac{d}{2} \rfloor - s} \sum_{\substack{L \in \mathcal{B}_\ell \\ (L,f)=1}} \sum_{\substack{N \in \mathbb{A}_{r-\ell}^+ \\ (N,f)=1}} \mu(N) \tilde{\Gamma}'_{f,L,s} \\ &\leq \sum_{\ell=0}^{\lfloor \frac{d}{2} \rfloor - s} (1 - q^{-1}) \binom{\lfloor d/2 \rfloor - s}{\ell} q^{r-\ell} q^{\ell+s} \leq 2^{\frac{d}{2}-s} q^{r+s}. \end{aligned}$$

Now, (3.21) follows from

$$\sum_{u \in \mathcal{I}_{g+1}} \left\{ \frac{u}{f} \right\} = \sum_{s=1}^{g+1} \sum_{u \in \mathcal{I}_{(g+1-s,s)}} \left\{ \frac{u}{f} \right\}.$$

□

4. PROOF OF THEOREM 2.4

In this section we give a proof of Theorem 2.4. In §4.1 and §4.2, we obtain several results on the contribution of squares and of non-squares, which will be used to calculate the first moment of L -functions for $\text{Re}(s) \geq \frac{1}{2}$ in §4.3, §4.4 and §4.5.

4.1. The Main Term: Contribution of squares. For a small $\varepsilon > 0$ and $s \in \bar{X}_\varepsilon$, let

$$A_g(s) = \begin{cases} \frac{P(1)}{\zeta_A(2)} \left(\left\lfloor \frac{g}{2} \right\rfloor + 1 + \frac{1}{\log q} \frac{P'}{P}(1) \right) & \text{if } s = \frac{1}{2}, \\ \frac{\zeta_A(2s)}{\zeta_A(2)} (P(2s) - q^{(1-2s)(\lfloor \frac{g}{2} \rfloor + 1)} P(1)) & \text{if } s \in X_\varepsilon, \\ \frac{\zeta_A(2s)}{\zeta_A(2)} P(2s) & \text{if } \text{Re}(s) \geq 1, \end{cases} \quad (4.1)$$

and

$$B_g(s) = \begin{cases} \frac{P(1)}{\zeta_A(2)} \left(\left\lfloor \frac{g-1}{2} \right\rfloor + 1 + \frac{1}{\log q} \frac{P'}{P}(1) \right) & \text{if } s = \frac{1}{2}, \\ \frac{\zeta_A(2s)}{\zeta_A(2)} q^{(1-2s)g} \left(q^{(2s-1)\lfloor \frac{g-1}{2} \rfloor} P(1) - q^{(1-2s)} P(2-2s) \right) & \text{if } s \in X_\varepsilon, \\ 0 & \text{if } \text{Re}(s) \geq 1. \end{cases} \quad (4.2)$$

Proposition 4.1. *For a small $\varepsilon > 0$ and $s \in \bar{X}_\varepsilon$, we have*

$$\begin{aligned} (1) \quad & \sum_{u \in \mathcal{I}_{g+1}} \sum_{n=0}^g q^{-sn} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f=\square}} \left\{ \frac{u}{f} \right\} = 2A_g(s) q^{2g+1} + O\left(gq^{\frac{3g}{2}}\right), \\ (2) \quad & q^{(1-2s)g} \sum_{u \in \mathcal{I}_{g+1}} \sum_{n=0}^{g-1} q^{(s-1)n} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f=\square}} \left\{ \frac{u}{f} \right\} = 2B_g(s) q^{2g+1} + O\left(gq^{\frac{3g}{2}}\right). \end{aligned}$$

Proof. (1) Write

$$\sum_{u \in \mathcal{I}_{g+1}} \sum_{n=0}^g q^{-sn} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f=\square}} \left\{ \frac{u}{f} \right\} = \sum_{r=0}^g \sum_{u \in \mathcal{I}_{(r,g+1-r)}} \sum_{n=0}^g q^{-sn} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f=\square}} \left\{ \frac{u}{f} \right\}.$$

Note that $\mathcal{I}_{(0,g+1)} = \mathcal{G}_{g+1}$. For $1 \leq r \leq g$ and $M \in \mathcal{B}_r$, let $\mathcal{I}_M = \{v + F : v \in \mathcal{F}_M \text{ and } F \in \mathcal{G}_{g+1-r}\}$. Then $\mathcal{I}_{(r,g+1-r)}$ is the disjoint union of the \mathcal{I}_M 's, where M runs over \mathcal{B}_r . Hence we have

$$\begin{aligned} & \sum_{r=0}^g \sum_{u \in \mathcal{I}_{(r,g+1-r)}} \sum_{n=0}^g q^{-sn} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f=\square}} \left\{ \frac{u}{f} \right\} \\ &= \sum_{l=0}^{\lfloor \frac{g}{2} \rfloor} q^{-2sl} \sum_{L \in \mathbb{A}_l^+} \sum_{F \in \mathcal{G}_{g+1}} \left\{ \frac{F}{L^2} \right\} + \sum_{l=0}^{\lfloor \frac{g}{2} \rfloor} q^{-2sl} \sum_{L \in \mathbb{A}_l^+} \sum_{r=1}^g \sum_{M \in \mathcal{B}_r} \sum_{u \in \mathcal{I}_M} \left\{ \frac{u}{L^2} \right\} \\ &= \sum_{l=0}^{\lfloor \frac{g}{2} \rfloor} q^{-2sl} \sum_{L \in \mathbb{A}_l^+} \sum_{F \in \mathcal{G}_{g+1}} 1 + \sum_{l=0}^{\lfloor \frac{g}{2} \rfloor} q^{-2sl} \sum_{L \in \mathbb{A}_l^+} \sum_{r=1}^g \sum_{\substack{M \in \mathcal{B}_r \\ (M,L)=1}} \sum_{u \in \mathcal{I}_M} 1. \end{aligned}$$

Since $\#\mathcal{G}_{g+1} = 2\zeta_{\mathbb{A}}(2)^{-1}q^{g+1}$, we have

$$\sum_{l=0}^{\lfloor \frac{g}{2} \rfloor} q^{-2sl} \sum_{L \in \mathbb{A}_l^+} \sum_{F \in \mathcal{G}_{g+1}} 1 = 2\zeta_{\mathbb{A}}(2)^{-1}q^{g+1} \sum_{l=0}^{\lfloor \frac{g}{2} \rfloor} q^{(1-2s)l} \ll gq^g.$$

Since $\#\mathcal{I}_M = \frac{2}{\zeta_{\mathbb{A}}(2)}q^{g+1-r}\Phi(\tilde{M})$, by (3.10), we have

$$\begin{aligned} & \sum_{l=0}^{\lfloor \frac{g}{2} \rfloor} q^{-2sl} \sum_{L \in \mathbb{A}_l^+} \sum_{r=1}^g \sum_{\substack{M \in \mathcal{B}_r \\ (M,L)=1}} \sum_{u \in \mathcal{I}_M} 1 = \frac{2}{\zeta_{\mathbb{A}}(2)}q^{g+1} \sum_{l=0}^{\lfloor \frac{g}{2} \rfloor} q^{-2sl} \sum_{L \in \mathbb{A}_l^+} \sum_{r=1}^g q^{-r} \sum_{\substack{\tilde{M} \in \mathbb{A}_r^+ \\ (\tilde{M},L)=1}} \Phi(\tilde{M}) \\ &= \frac{2}{\zeta_{\mathbb{A}}(2)}(q^g - 1)q^{g+1}J_g(s) + O\left(q^{\frac{3g}{2}} \sum_{l=0}^{\lfloor \frac{g}{2} \rfloor} q^{(1-2s)l}\right), \quad (4.3) \end{aligned}$$

where $J_g(s)$ is given in (3.16). The error term in (4.3) is $\ll gq^{\frac{3g}{2}}$. Now, the result follows from Lemma 3.6.

(2) By a similar process as in the proof of (1), we have

$$\begin{aligned} & q^{(1-2s)g} \sum_{u \in \mathcal{I}_{g+1}} \sum_{n=0}^{g-1} q^{(s-1)n} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f=\square}} \left\{ \frac{u}{f} \right\} = q^{(1-2s)g} \sum_{F \in \mathcal{G}_{g+1}} \sum_{n=0}^{g-1} q^{(s-1)n} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f=\square}} \left\{ \frac{F}{f} \right\} \\ & \quad + q^{(1-2s)g} \sum_{r=0}^g \sum_{u \in \mathcal{I}_{(r,g+1-r)}} \sum_{n=0}^{g-1} q^{(s-1)n} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f=\square}} \left\{ \frac{u}{f} \right\} \\ &= q^{(1-2s)g} \sum_{l=0}^{\lfloor \frac{g-1}{2} \rfloor} q^{2(s-1)l} \sum_{L \in \mathbb{A}_l^+} \sum_{r=1}^g \sum_{\substack{\tilde{M} \in \mathbb{A}_r^+ \\ (\tilde{M},L)=1}} \Phi(\tilde{M}) + O\left(gq^{(\frac{3}{2}-s)g}\right) \\ &= \frac{2}{\zeta_{\mathbb{A}}(2)}(q^g - 1)q^{(2-2s)g+1}\tilde{J}_{g-1}(s) + O\left(gq^{\frac{3g}{2}}\right), \end{aligned}$$

where $\tilde{J}_{g-1}(s)$ is given in (3.17). The result follows from Lemma 3.7. \square

Proposition 4.2. *For a small $\varepsilon > 0$ and $s \in \bar{X}_\varepsilon$, we have*

$$\begin{aligned}
(1) \quad & \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^g (\pm 1)^n q^{-sn} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f = \square}} \left\{ \frac{u}{f} \right\} = A_g(s) q^{2g+2} + O\left(g q^{\frac{3g}{2}}\right), \\
(2) \quad & q^{(1-2s)g} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} (\pm 1)^n q^{(s-1)n} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f = \square}} \left\{ \frac{u}{f} \right\} = B_g(s) q^{2g+2} + O\left(g q^{\frac{3g}{2}}\right).
\end{aligned}$$

Proof. (1) For each $M \in \mathcal{B}_{g+1}$, let \mathcal{F}_M be the set of rational functions $u \in \mathcal{F}_{g+1}$ whose denominator is M . Then \mathcal{F}_{g+1} is a disjoint union of the \mathcal{F}_M 's, where M runs over \mathcal{B}_{g+1} . Hence, we can write

$$\begin{aligned}
\sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^g (\pm 1)^n q^{-sn} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f = \square}} \left\{ \frac{u}{f} \right\} &= \sum_{n=0}^g (\pm 1)^n q^{-sn} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f = \square}} \sum_{M \in \mathcal{B}_{g+1}} \sum_{u \in \mathcal{F}_M} \left\{ \frac{u}{f} \right\} \\
&= \sum_{l=0}^{\lfloor \frac{g}{2} \rfloor} q^{-2sl} \sum_{L \in \mathbb{A}_l^+} \sum_{M \in \mathcal{B}_{g+1}} \sum_{u \in \mathcal{F}_M} \left\{ \frac{u}{L^2} \right\} \\
&= \sum_{l=0}^{\lfloor \frac{g}{2} \rfloor} q^{-2sl} \sum_{L \in \mathbb{A}_l^+} \sum_{\substack{M \in \mathcal{B}_{g+1} \\ (M, L)=1}} \sum_{u \in \mathcal{F}_M} 1.
\end{aligned}$$

Since $\#\mathcal{F}_M = \Phi(\tilde{M})$, by (3.10), we have

$$\begin{aligned}
\sum_{l=0}^{\lfloor \frac{g}{2} \rfloor} q^{-2sl} \sum_{L \in \mathbb{A}_l^+} \sum_{\substack{M \in \mathcal{B}_{g+1} \\ (M, L)=1}} \sum_{u \in \mathcal{F}_M} 1 &= \sum_{l=0}^{\lfloor \frac{g}{2} \rfloor} q^{-2sl} \sum_{L \in \mathbb{A}_l^+} \sum_{\substack{\tilde{M} \in \mathbb{A}_{g+1}^+ \\ (\tilde{M}, L)=1}} \Phi(\tilde{M}) \\
&= \frac{1}{\zeta_{\mathbb{A}}(2)} q^{2g+2} J_g(s) + O\left(q^{\frac{3g}{2}} \sum_{l=0}^{\lfloor \frac{g}{2} \rfloor} q^{(1-2s)l}\right). \tag{4.4}
\end{aligned}$$

The error term in (4.4) is $\ll g q^{\frac{3g}{2}}$, and the result follows from Lemma 3.6.

(2) By a similar process as in the proof of (1), we have

$$\begin{aligned}
& q^{(1-2s)g} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} (\pm 1)^n q^{(s-1)n} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f = \square}} \left\{ \frac{u}{f} \right\} \\
&= q^{(1-2s)g} \sum_{n=0}^{g-1} (\pm 1)^n q^{(s-1)n} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f = \square}} \sum_{M \in \mathcal{B}_{g+1}} \sum_{u \in \mathcal{F}_M} \left\{ \frac{u}{f} \right\} \\
&= q^{(1-2s)g} \sum_{l=0}^{\lfloor \frac{g-1}{2} \rfloor} q^{2(s-1)l} \sum_{L \in \mathbb{A}_l^+} \sum_{\substack{\tilde{M} \in \mathbb{A}_{g+1}^+ \\ (\tilde{M}, L)=1}} \Phi(\tilde{M}) \\
&= \frac{1}{\zeta_{\mathbb{A}}(2)} \tilde{J}_{g-1}(s) q^{(3-2s)g+2} + O\left(q^{(\frac{5}{2}-2s)g} \sum_{l=0}^{\lfloor \frac{g-1}{2} \rfloor} q^{(2s-1)l}\right), \tag{4.5}
\end{aligned}$$

where the error term in (4.5) is $\ll g q^{\frac{3g}{2}}$. Now, the result follows from Lemma 3.7. \square

Proposition 4.3. *Let $h \in \{g-1, g\}$. For $s \in \mathbb{C}$ with $\operatorname{Re}(s) \geq \frac{1}{2}$, we have*

$$q^{-(h+1)s} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^h (\pm 1)^n \sum_{\substack{f \in \mathbb{A}_n^+ \\ f=\square}} \left\{ \frac{u}{f} \right\} = P(1) q^{2g+2-(h+1)s+\lceil \frac{h}{2} \rceil} + O\left(g q^{(2-s)g}\right).$$

Proof. By a similar process as in the proof of (1) of Proposition 4.2, we have

$$\begin{aligned} q^{-(h+1)s} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^h (\pm 1)^n \sum_{\substack{f \in \mathbb{A}_n^+ \\ f=\square}} \left\{ \frac{u}{f} \right\} &= q^{-(h+1)s} \sum_{n=0}^h (\pm 1)^n \sum_{\substack{f \in \mathbb{A}_n^+ \\ f=\square}} \sum_{M \in \mathcal{B}_{g+1}} \sum_{u \in \mathcal{F}_M} \left\{ \frac{u}{f} \right\} \\ &= q^{-(h+1)s} \sum_{l=0}^{\lceil \frac{h}{2} \rceil} \sum_{L \in \mathbb{A}_l^+} \sum_{M \in \mathcal{B}_{g+1}} \sum_{u \in \mathcal{F}_M} \left\{ \frac{u}{L^2} \right\} \\ &= q^{-(h+1)s} \sum_{l=0}^{\lceil \frac{h}{2} \rceil} \sum_{L \in \mathbb{A}_l^+} \sum_{\substack{\tilde{M} \in \mathbb{A}_{g+1}^+ \\ (\tilde{M}, L)=1}} \Phi(\tilde{M}) \\ &= \frac{1}{\zeta_{\mathbb{A}}(2)} q^{2g+2-(h+1)s} \tilde{J}_h(1) + O\left(q^{(2-s)g}\right). \end{aligned}$$

Now, the result follows from Lemma 3.7. □

Proposition 4.4. *For $s \in \mathbb{C}$ with $\operatorname{Re}(s) \geq \frac{1}{2}$, we have*

$$q^{-g} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} (g-n) \sum_{\substack{f \in \mathbb{A}_n^+ \\ f=\square}} \left\{ \frac{u}{f} \right\} = O\left(g q^{\frac{3g}{2}}\right).$$

Proof. By a similar process as in the proof of Proposition 4.2 (1), and by Corollary 3.4, we have

$$\begin{aligned} q^{-g} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} (g-n) \sum_{\substack{f \in \mathbb{A}_n^+ \\ f=\square}} \left\{ \frac{u}{f} \right\} &= q^{-g} \sum_{n=0}^{g-1} (g-n) \sum_{\substack{f \in \mathbb{A}_n^+ \\ f=\square}} \sum_{M \in \mathcal{B}_{g+1}} \sum_{u \in \mathcal{F}_M} \left\{ \frac{u}{f} \right\} \\ &= q^{-g} \sum_{l=0}^{\lceil \frac{g-1}{2} \rceil} (g-2l) \sum_{L \in \mathbb{A}_l^+} \sum_{M \in \mathcal{B}_{g+1}} \sum_{u \in \mathcal{F}_M} \left\{ \frac{u}{L^2} \right\} \\ &= q^{-g} \sum_{l=0}^{\lceil \frac{g-1}{2} \rceil} (g-2l) \sum_{L \in \mathbb{A}_l^+} \sum_{\substack{\tilde{M} \in \mathbb{A}_{g+1}^+ \\ (\tilde{M}, L)=1}} \Phi(\tilde{M}) \\ &= \frac{1}{\zeta_{\mathbb{A}}(2)} q^{g+1} \sum_{l=0}^{\lceil \frac{g-1}{2} \rceil} (g-2l) q^l \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq l}} \frac{\mu(D)}{|D|} \prod_{P|D} \frac{1}{|P|+1} + O(q^g). \end{aligned}$$

Moreover, we have

$$\begin{aligned}
q^{g+1} \sum_{l=0}^{\lfloor \frac{g-1}{2} \rfloor} (g-2l)q^l \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq l}} \frac{\mu(D)}{|D|} \prod_{P|D} \frac{1}{|P|+1} &\ll q^g \sum_{l=0}^{\lfloor \frac{g-1}{2} \rfloor} (g-2l)q^l \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq l}} \frac{\mu^2(D)}{|D|} \prod_{P|D} \frac{1}{|P|} \\
&\ll q^g \sum_{l=0}^{\lfloor \frac{g-1}{2} \rfloor} (g-2l)q^l \sum_{\substack{D \in \mathbb{A}^+ \\ \deg(D) \leq l}} |D|^{-2} \\
&\ll q^g \sum_{l=0}^{\lfloor \frac{g-1}{2} \rfloor} l(g-2l)q^l \ll gq^{\frac{3g}{2}}.
\end{aligned}$$

Hence we get the result. \square

4.2. The Error Term: Contributions of non-squares. Let $s_0 = 1 + \frac{1}{2} \log_q 2$.

Proposition 4.5. *For $s \in \mathbb{C}$ with $\operatorname{Re}(s) \geq \frac{1}{2}$, we have*

$$\begin{aligned}
(1) \quad \sum_{u \in \mathcal{I}_{g+1}} \sum_{n=0}^g q^{-sn} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \left\{ \frac{u}{f} \right\} &\ll \begin{cases} g2^{\frac{g}{2}} q^{(2-s)g} & \text{if } \operatorname{Re}(s) < s_0, \\ g^2 q^g & \text{if } \operatorname{Re}(s) \geq s_0, \end{cases} \\
(2) \quad q^{(1-2s)g} \sum_{u \in \mathcal{I}_{g+1}} \sum_{n=0}^{g-1} q^{(s-1)n} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \left\{ \frac{u}{f} \right\} &\ll g2^{\frac{g}{2}} q^{(2-s)g}.
\end{aligned}$$

Proof. (1) By (3.21), we have

$$\begin{aligned}
\sum_{u \in \mathcal{I}_{g+1}} \sum_{n=0}^g q^{-sn} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \left\{ \frac{u}{f} \right\} &\ll \sum_{n=0}^g q^{-sn} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \left| \sum_{u \in \mathcal{I}_{g+1}} \left\{ \frac{u}{f} \right\} \right| \\
&\ll gq^g \sum_{n=0}^g (2^{1/2} q^{1-s})^n \\
&\ll \begin{cases} g2^{\frac{g}{2}} q^{(2-s)g} & \text{if } \operatorname{Re}(s) < s_0, \\ g^2 q^g & \text{if } \operatorname{Re}(s) \geq s_0. \end{cases}
\end{aligned}$$

The proof of (2) is similar to that of (1) and there is no novelty involved. \square

Proposition 4.6. *For $s \in \mathbb{C}$ with $\operatorname{Re}(s) \geq \frac{1}{2}$, we have*

$$\begin{aligned}
(1) \quad \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^g (\pm 1)^n q^{-sn} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \left\{ \frac{u}{f} \right\} &\ll \begin{cases} 2^{\frac{g}{2}} q^{(2-s)g} & \text{if } \operatorname{Re}(s) < s_0, \\ gq^g & \text{if } \operatorname{Re}(s) \geq s_0, \end{cases} \\
(2) \quad q^{(1-2s)g} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} (\pm 1)^n q^{(s-1)n} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \left\{ \frac{u}{f} \right\} &\ll 2^{\frac{g}{2}} q^{(2-s)g}, \\
(3) \quad q^{-(h+1)s} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^h (\pm 1)^n \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \left\{ \frac{u}{f} \right\} &\ll 2^{\frac{g}{2}} q^{(2-s)g}, \text{ where } h \in \{g-1, g\},
\end{aligned}$$

$$(4) \quad q^{-g} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} (g-n) \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \left\{ \frac{u}{f} \right\} \ll 2^{\frac{g}{2}} q^g.$$

Proof. (1) By Proposition 3.15, we have

$$\begin{aligned} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^g (\pm 1)^n q^{-sn} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \left\{ \frac{u}{f} \right\} &\ll \sum_{n=0}^g q^{-sn} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \left| \sum_{u \in \mathcal{F}_{g+1}} \left\{ \frac{u}{f} \right\} \right| \\ &\ll q^g \sum_{n=0}^g (2^{\frac{1}{2}} q^{1-s})^n \\ &\ll \begin{cases} 2^{\frac{g}{2}} q^{(2-s)g} & \text{if } \operatorname{Re}(s) < s_0, \\ gq^g & \text{if } \operatorname{Re}(s) \geq s_0. \end{cases} \end{aligned}$$

The proofs of (2), (3) and (4) are similar to that of (1) and there is no novelty involved. \square

Remark 4.7. Let $s_1 = \frac{1}{2}(1 + \log_q 2)$. For $s \in \mathbb{C}$ with $\operatorname{Re}(s) \geq \frac{1}{2}$, we have

$$q^{\frac{3g}{2}} \ll 2^{\frac{g}{2}} q^{(2-s)g} \quad \text{if } \operatorname{Re}(s) < s_1$$

and

$$q^{\frac{3g}{2}} \gg 2^{\frac{g}{2}} q^{(2-s)g} \quad \text{if } \operatorname{Re}(s) \geq s_1.$$

4.3. Proof of Theorem 2.4 (1). By Lemma 3.1 (1), we have

$$\sum_{u \in \mathcal{I}_{g+1}} L(s, \chi_u) = \sum_{u \in \mathcal{I}_{g+1}} \sum_{n=0}^g q^{-sn} \sum_{f \in \mathbb{A}_n^+} \chi_u(f) + q^{(1-2s)g} \sum_{u \in \mathcal{I}_{g+1}} \sum_{n=0}^{g-1} q^{(s-1)n} \sum_{f \in \mathbb{A}_n^+} \chi_u(f). \quad (4.6)$$

For a small $\varepsilon > 0$ and $s \in \bar{X}_\varepsilon$, by Propositions 4.1, 4.5 and Remark 4.7, we have

$$\begin{aligned} \sum_{u \in \mathcal{I}_{g+1}} \sum_{n=0}^g q^{-sn} \sum_{f \in \mathbb{A}_n^+} \chi_u(f) &= \sum_{u \in \mathcal{I}_{g+1}} \sum_{n=0}^g q^{-sn} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f = \square}} \chi_u(f) + \sum_{u \in \mathcal{I}_{g+1}} \sum_{n=0}^g q^{-sn} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \chi_u(f) \\ &= 2A_g(s)q^{2g+1} + \begin{cases} O(g2^{\frac{g}{2}} q^{(2-s)g}) & \text{if } \operatorname{Re}(s) < s_1, \\ O(gq^{\frac{3g}{2}}) & \text{if } \operatorname{Re}(s) \geq s_1, \end{cases} \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} q^{(1-2s)g} \sum_{u \in \mathcal{I}_{g+1}} \sum_{n=0}^{g-1} q^{(s-1)n} \sum_{f \in \mathbb{A}_n^+} \chi_u(f) &= q^{(1-2s)g} \sum_{u \in \mathcal{I}_{g+1}} \sum_{n=0}^{g-1} q^{(s-1)n} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f = \square}} \chi_u(f) \\ &\quad + q^{(1-2s)g} \sum_{u \in \mathcal{I}_{g+1}} \sum_{n=0}^{g-1} q^{(s-1)n} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \chi_u(f) \\ &= 2B_g(s)q^{2g+1} + \begin{cases} O(g2^{\frac{g}{2}} q^{(2-s)g}) & \text{if } \operatorname{Re}(s) < s_1, \\ O(gq^{\frac{3g}{2}}) & \text{if } \operatorname{Re}(s) \geq s_1. \end{cases} \end{aligned} \quad (4.8)$$

By inserting (4.7) and (4.8) into (4.6), we have

$$\sum_{u \in \mathcal{I}_{g+1}} L(s, \chi_u) = 2(A_g(s) + B_g(s))q^{2g+1} + \begin{cases} O(g2^{\frac{g}{2}} q^{(2-s)g}) & \text{if } \operatorname{Re}(s) < s_1, \\ O(gq^{\frac{3g}{2}}) & \text{if } \operatorname{Re}(s) \geq s_1, \end{cases}$$

where $A_g(s) + B_g(s)$ equals to

$$\alpha_g(s) = \begin{cases} \frac{P(1)}{\zeta_{\mathbb{A}}(2)} \left(g + 1 + \frac{2}{\log q} \frac{P'}{P}(1) \right) & \text{if } s = \frac{1}{2}, \\ \frac{\zeta_{\mathbb{A}}(2s)}{\zeta_{\mathbb{A}}(2)} \left\{ P(2s) - q^{(1-2s)(g+1)} P(2-2s) \right. \\ \quad \left. + P(1) \left(q^{(1-2s)(g - [\frac{g-1}{2}])} - q^{(1-2s)([\frac{g}{2}]+1)} \right) \right\} & \text{if } s \in X_{\varepsilon}, \\ \frac{\zeta_{\mathbb{A}}(2s)}{\zeta_{\mathbb{A}}(2)} P(2s) & \text{if } \operatorname{Re}(s) \geq 1. \end{cases}$$

4.4. Proof of Theorem 2.4 (2). By Lemma 3.1 (2), we have

$$\begin{aligned} \sum_{u \in \mathcal{F}_{g+1}} L(s, \chi_u) &= \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^g q^{-sn} \sum_{f \in \mathbb{A}_n^+} \chi_u(f) \\ &\quad - q^{-(g+1)s} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^g \sum_{f \in \mathbb{A}_n^+} \chi_u(f) + \sum_{u \in \mathcal{F}_{g+1}} H_u(s), \end{aligned} \quad (4.9)$$

where

$$\sum_{u \in \mathcal{F}_{g+1}} H_u(1) = \zeta_{\mathbb{A}}(2)^{-1} q^{-g} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} (g-n) \sum_{f \in \mathbb{A}_n^+} \left\{ \frac{u}{f} \right\}$$

and, for $s \neq 1$,

$$\sum_{u \in \mathcal{F}_{g+1}} H_u(s) = q^{(1-2s)g} \eta(s) \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} q^{(s-1)n} \sum_{f \in \mathbb{A}_n^+} \left\{ \frac{u}{f} \right\} - q^{-sg} \eta(s) \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} \sum_{f \in \mathbb{A}_n^+} \left\{ \frac{u}{f} \right\}$$

with $\eta(s) = \frac{\zeta_{\mathbb{A}}(2-s)}{\zeta_{\mathbb{A}}(1+s)}$. For a small $\varepsilon > 0$ and $s \in \bar{X}_{\varepsilon}$, by Propositions 4.2, 4.3, 4.6 and Remark 4.7, we have

$$\begin{aligned} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^g q^{-sn} \sum_{f \in \mathbb{A}_n^+} \chi_u(f) &= \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^g q^{-sn} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f = \square}} \chi_u(f) + \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^g q^{-sn} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \chi_u(f) \\ &= A_g(s) q^{2g+2} + \begin{cases} O(2^{\frac{g}{2}} q^{(2-s)g}) & \text{if } \operatorname{Re}(s) < s_1, \\ O(g q^{\frac{3g}{2}}) & \text{if } \operatorname{Re}(s) \geq s_1, \end{cases} \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} q^{-(g+1)s} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^g \sum_{f \in \mathbb{A}_n^+} \chi_u(f) &= q^{-(g+1)s} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^g \sum_{\substack{f \in \mathbb{A}_n^+ \\ f = \square}} \chi_u(f) \\ &\quad + q^{-(g+1)s} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^g \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \chi_u(f) \\ &= P(1) q^{(2-s)g + [\frac{g}{2}] + 2 - s} + \begin{cases} O(2^{\frac{g}{2}} q^{(2-s)g}) & \text{if } \operatorname{Re}(s) < s_1, \\ O(g q^{\frac{3g}{2}}) & \text{if } \operatorname{Re}(s) \geq s_1. \end{cases} \end{aligned} \quad (4.11)$$

If $s = 1$, by Propositions 4.4 and 4.6 (4), we have

$$\begin{aligned} \sum_{u \in \mathcal{F}_{g+1}} H_u(1) &= \zeta_{\mathbb{A}}(2)^{-1} q^{-g} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} (g-n) \sum_{\substack{f \in \mathbb{A}_n^+ \\ f=\square}} \chi_u(f) \\ &\quad + \zeta_{\mathbb{A}}(2)^{-1} q^{-g} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} (g-n) \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \chi_u(f) = O\left(gq^{\frac{3g}{2}}\right). \end{aligned} \quad (4.12)$$

By inserting (4.10), (4.11) with $s = 1$ and (4.12) into (4.9), we have

$$\sum_{u \in \mathcal{F}_{g+1}} L(1, \chi_u) = P(2)q^{2g+2} + O\left(gq^{\frac{3g}{2}}\right). \quad (4.13)$$

Now, consider the case $s \neq 1$. Let $\delta > 0$ be arbitrary small. For $s \in \bar{X}_\varepsilon$ with $|s - 1| > \delta$, since $\eta(s)$ is bounded, by Propositions 4.2 and 4.6, we have

$$\begin{aligned} q^{(1-2s)g} \eta(s) \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} q^{(s-1)n} \sum_{f \in \mathbb{A}_n^+} \left\{ \frac{u}{f} \right\} &= q^{(1-2s)g} \eta(s) \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} q^{(s-1)n} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f=\square}} \left\{ \frac{u}{f} \right\} \\ &\quad + q^{(1-2s)g} \eta(s) \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} q^{(s-1)n} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \left\{ \frac{u}{f} \right\} \\ &= \eta(s) B_g(s) q^{2g+2} + \begin{cases} O(2^{\frac{g}{2}} q^{(2-s)g}) & \text{if } \operatorname{Re}(s) < s_1, \\ O(gq^{\frac{3g}{2}}) & \text{if } \operatorname{Re}(s) \geq s_1, \end{cases} \end{aligned} \quad (4.14)$$

and, by Propositions 4.3 and 4.6,

$$\begin{aligned} q^{-sg} \eta(s) \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} \sum_{f \in \mathbb{A}_n^+} \left\{ \frac{u}{f} \right\} &= q^{-sg} \eta(s) \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f=\square}} \left\{ \frac{u}{f} \right\} \\ &\quad + q^{-sg} \eta(s) \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \left\{ \frac{u}{f} \right\} \\ &= \eta(s) P(1) q^{(2-s)g + [\frac{g-1}{2}] + 2} + \begin{cases} O(2^{\frac{g}{2}} q^{(2-s)g}) & \text{if } \operatorname{Re}(s) < s_1, \\ O(gq^{\frac{3g}{2}}) & \text{if } \operatorname{Re}(s) \geq s_1. \end{cases} \end{aligned} \quad (4.15)$$

By inserting (4.10), (4.11), (4.14) and (4.15) into (4.9), we have

$$\sum_{u \in \mathcal{F}_{g+1}} L(s, \chi_u) = \beta_g(s) q^{2g+2} + \begin{cases} O(2^{\frac{g}{2}} q^{(2-s)g}) & \text{if } \operatorname{Re}(s) < s_1, \\ O(gq^{\frac{3g}{2}}) & \text{if } \operatorname{Re}(s) \geq s_1. \end{cases} \quad (4.16)$$

4.5. Proof of Theorem 2.4 (3). Note that $\chi_u(f) = (-1)^{\deg(f)}\chi_v(f)$ for $u = v + \xi \in \mathcal{F}'_{g+1}$ with $v \in \mathcal{F}_{g+1}$. Then, by Lemma 3.1 (3), we have

$$\begin{aligned} \sum_{u \in \mathcal{F}'_{g+1}} L(s, \chi_u) &= \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^g (-1)^n q^{-sn} \sum_{f \in \mathbb{A}_n^+} \chi_u(f) + (-1)^g q^{-(g+1)s} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^g \sum_{f \in \mathbb{A}_n^+} \chi_u(f) \\ &\quad + \nu(s) q^{(1-2s)g} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} (-1)^n q^{(s-1)n} \sum_{f \in \mathbb{A}_n^+} \chi_u(f) \\ &\quad + (-1)^{g+1} \nu(s) q^{-sg} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} \sum_{f \in \mathbb{A}_n^+} \chi_u(f) \end{aligned} \quad (4.17)$$

with $\nu(s) = \frac{1+q^{-s}}{1+q^{s-1}}$. For a small $\varepsilon > 0$ and $s \in \bar{X}_\varepsilon$, by Propositions 4.2 and 4.6, we have

$$\begin{aligned} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^g (-1)^n q^{-sn} \sum_{f \in \mathbb{A}_n^+} \chi_u(f) &= \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^g (-1)^n q^{-sn} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f = \square}} \chi_u(f) \\ &\quad + \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^g (-1)^n q^{-sn} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \chi_u(f) \\ &= A_g(s) q^{2g+2} + \begin{cases} O(2^{\frac{g}{2}} q^{(2-s)g}) & \text{if } \operatorname{Re}(s) < s_1, \\ O(g q^{\frac{3g}{2}}) & \text{if } \operatorname{Re}(s) \geq s_1, \end{cases} \end{aligned} \quad (4.18)$$

and

$$\begin{aligned} &\nu(s) q^{(1-2s)g} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} (-1)^n q^{(s-1)n} \sum_{f \in \mathbb{A}_n^+} \chi_u(f) \\ &= \nu(s) q^{(1-2s)g} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} (-1)^n q^{(s-1)n} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f = \square}} \chi_u(f) \\ &\quad + \nu(s) q^{(1-2s)g} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} (-1)^n q^{(s-1)n} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \chi_u(f) \\ &= \nu(s) B_g(s) q^{2g+2} + \begin{cases} O(2^{\frac{g}{2}} q^{(2-s)g}) & \text{if } \operatorname{Re}(s) < s_1, \\ O(g q^{\frac{3g}{2}}) & \text{if } \operatorname{Re}(s) \geq s_1. \end{cases} \end{aligned} \quad (4.19)$$

By Propositions 4.3 and 4.6, we have

$$\begin{aligned} (-1)^g q^{-(g+1)s} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^g \sum_{f \in \mathbb{A}_n^+} \chi_u(f) &= (-1)^g q^{-(g+1)s} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^g \sum_{\substack{f \in \mathbb{A}_n^+ \\ f = \square}} \chi_u(f) \\ &\quad + (-1)^g q^{-(g+1)s} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^g \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \chi_u(f) \\ &= (-1)^g P(1) q^{(2-s)g + [\frac{g}{2}] + 2 - s} + \begin{cases} O(2^{\frac{g}{2}} q^{(2-s)g}) & \text{if } \operatorname{Re}(s) < s_1, \\ O(g q^{\frac{3g}{2}}) & \text{if } \operatorname{Re}(s) \geq s_1, \end{cases} \end{aligned} \quad (4.20)$$

and

$$\begin{aligned}
(-1)^{g+1} \nu(s) q^{-sg} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} \sum_{f \in \mathbb{A}_n^+} \chi_u(f) &= (-1)^{g+1} \nu(s) q^{-sg} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f=\square}} \chi_u(f) \\
&\quad + (-1)^{g+1} \nu(s) q^{-sg} \sum_{u \in \mathcal{F}_{g+1}} \sum_{n=0}^{g-1} \sum_{\substack{f \in \mathbb{A}_n^+ \\ f \neq \square}} \chi_u(f) \\
&= (-1)^{g+1} \nu(s) P(1) q^{(2-s)g + [\frac{g-1}{2}] + 2} + \begin{cases} O(g 2^{\frac{g}{2}} q^{(2-s)g}) & \text{if } \operatorname{Re}(s) < s_1, \\ O(g q^{\frac{3g}{2}}) & \text{if } \operatorname{Re}(s) \geq s_1. \end{cases} \quad (4.21)
\end{aligned}$$

By inserting (4.18), (4.19), (4.20) and (4.21) into (4.17), we have

$$\sum_{u \in \mathcal{F}'_{g+1}} L(s, \chi_u) = \gamma_g(s) q^{2g+2} + \begin{cases} O(2^{\frac{g}{2}} q^{(2-s)g}) & \text{if } \operatorname{Re}(s) < s_1, \\ O(g q^{\frac{3g}{2}}) & \text{if } \operatorname{Re}(s) \geq s_1. \end{cases}$$

Remark 4.8. *With the results in §3.1 and §3.2, we can modify the proofs in §4 by replacing $u \in \mathcal{I}_{g+1}$ (resp. \mathcal{F}_{g+1} , \mathcal{F}'_{g+1}) by $D \in \mathcal{H}_{2g+1}$ (resp. \mathcal{H}_{2g+2} , $\gamma \mathcal{H}_{2g+2}$), and $\{\frac{u}{f}\}$ by $(\frac{D}{f})$ to show that Theorem 2.4 is also true in odd characteristic. Here \mathcal{H}_n is the set of all monic square-free polynomials of degree n , and γ is a generator of \mathbb{F}_q^\times .*

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Department of Mathematics, KAIST, Daejeon 305-701, Korea

E-mail address: `shbae@kaist.ac.kr`

Department of Mathematics Education, Chungbuk National University, Cheongju 361-763, Korea

E-mail address: `hyjung@chungbuk.ac.kr`